Modulus on graphs as a generalization of standard graph theoretic quantities

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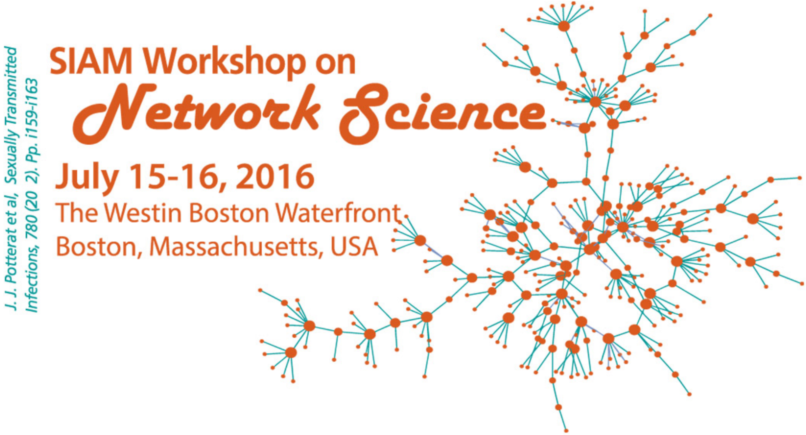

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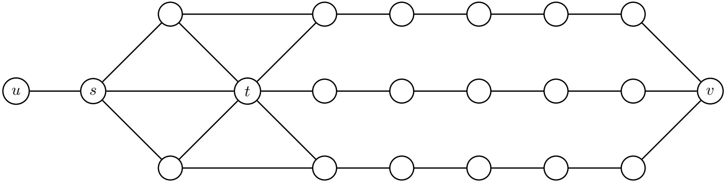
How "easily" can individual s transmit a disease to individual t?



- Shortest path?

$$\ell_{u,s} = 1, \quad \ell_{s,t} = 1, \quad \ell_{t,v} = 6$$

- Number of paths (maximum flow)?

$$f_{u,s} = 1, \quad f_{s,t} = 3, \quad f_{t,v} = 3$$

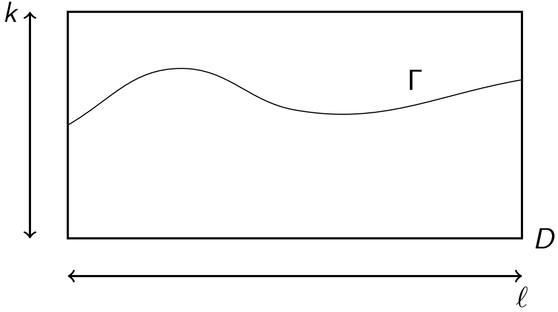
- Resistance distance (resistance in the resistor network)?

$$\Omega_{u,s} = 1, \quad \Omega_{s,t} = 0.454, \quad \Omega_{t,v} = 1.918$$

$$
\begin{tikzcd}
\text{Modulus of a family of curves} \arrow[d] & \{\text{Domains in } \mathbb{C}\} \arrow[l] \\
p\text{-Modulus of a family of } \textbf{walks} & \{\text{Graphs}\} \arrow[u]
\end{tikzcd}
$$

Example:

$$D = \{z \mid 0 \leq \mathrm{Re}(z) \leq \ell,\ 0 \leq \mathrm{Im}(z) \leq k\}$$
$$\Gamma = \{\text{curves from the left side of } D \text{ to the right}\}$$



How "constrained" is someone traveling Γ?

Moduli of Families of Curves for Conformal and Quasiconformal Mappings
(Springer–Verlag, 2015)

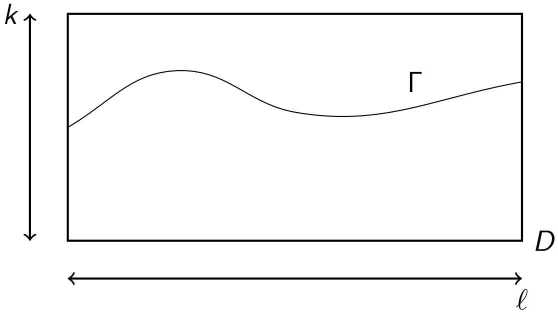
Let

- $D \subseteq \overline{\mathbb{C}}$ be a domain
- Γ be a family of paths $\gamma \subset D$
- $\mathrm{P} \subset L^2(D)$ be the integrable real non-negative functions

Define

- the ρ-length $\ell_\rho(\gamma) := \int_\gamma \rho(z)|dz|$
- admissible functions $A(\Gamma) = \{\rho \in \mathrm{P} \mid \forall\ \gamma \in \Gamma,\ \ell_\rho(\gamma) \geq 1\}$
- the ρ-area $\mathcal{A}_\rho(D) := \iint_D \rho(z)^2 dxdy$
- the **modulus** of Γ in D

$$\mathrm{Mod}(D, \Gamma) := \inf_{\rho \in A(\Gamma)} \mathcal{A}_\rho(D)$$



Exercise: $\operatorname{Mod}(D, \Gamma) = \dfrac{k}{\ell}$.

$\leq$: Use $\rho^*(z) \equiv \dfrac{1}{\ell}$ and check that $\rho^* \in A(\Gamma)$.

$\geq$: Verify $\mathcal{A}_\rho(D) \geq k$ in order to get

$$\iint_D \left(\frac{1}{\ell} - \rho(z)\right)^2 dxdy \geq 0.$$

Idea: $\mathrm{Mod}(D, \Gamma)$ measures

- the "richness" per "distance" of Γ
- the "ease" or absence of "constraint" along Γ

Note: the **extremal length** of Γ in D

$$\frac{1}{\mathrm{Mod}(D, \Gamma)} = \sup_{\rho} \frac{(\inf_{\gamma \in \Gamma} \ell_\rho(\gamma))^2}{\mathcal{A}_\rho(D)}$$

then measures the constraint or difficulty of Γ

Acta Mathematica (1950) **83**(1): 101–129

Let

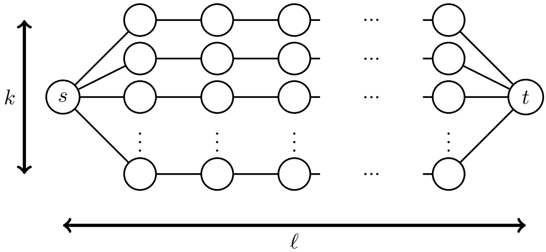
- $G = (V, E, \sigma)$ be a (directed, weighted) graph
- Γ be a family of walks $\gamma = e_1 e_2 \cdots e_r$ on G
- $\mathrm{P} \subseteq \mathbb{R}^E$ be the real non-negative edge densities
- $1 \leq p < \infty$

Define

- the ρ-length $\ell_\rho(\gamma) := \sum_{i=1}^r \rho(e_i)$
- admissible densities $A(\Gamma) := \{\rho \in \mathrm{P} \mid \forall \gamma \in \Gamma,\ \ell_\rho(\gamma) \geq 1\}$
- the p-energy $\mathcal{E}_\rho^{(p)}(G) := \sum_{e \in E} \sigma(e)|\rho(e)|^p$
- the p-**modulus** of Γ on G

$$\mathrm{Mod}_p(G, \Gamma) := \inf_{\rho \in A(\Gamma)} \mathcal{E}_\rho^{(p)}(G)$$

How “constrained” is the transmission from s to t?



Exercise: $\mathrm{Mod}_p(G,\Gamma) = \dfrac{k}{\ell^{p-1}}$.

$\leq$: Use $\rho^*(e) \equiv \dfrac{1}{\ell}$ and check that $\rho^* \in A(\Gamma)$.

$\geq$: Check that ρ^* is extremal, i.e. that
$\forall \rho' \in A(\Gamma),\ \mathcal{E}^{(p)}_{\rho'}(G) \geq \mathcal{E}^{(p)}_{\rho^*}(G)$.

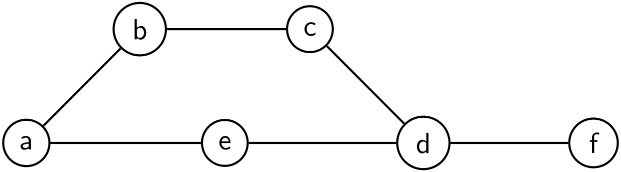
Connecting Family Theorem (2/3): Let

- $G = (V, E, \sigma)$ be undirected
- $\Gamma(s, t)$ be the family of walks from s to t

Then

- $\mathrm{Mod}_1(G, \Gamma(s, t)) =$ the maximum flow from s to t.
- $\mathrm{Mod}_2(G, \Gamma(s, t)) =$ the effective conductance btw s and t.

Example:



Lemma: Let

$$\begin{aligned} A'(\Gamma) &:= \{\rho : E \to \mathbb{R} \mid \ell_\rho(\Gamma) \geq 1\} \\ \supset A(\Gamma) &:= \{\rho : E \to \mathbb{R}_{\geq 0} \mid \ell_\rho(\Gamma) \geq 1\} \\ \supset A^*(\Gamma) &:= \{\rho : E \to [0,1] \mid \ell_\rho(\Gamma) \geq 1\} \end{aligned}$$

For all $p \geq 0$,

$$\inf_{\rho \in A'(\Gamma)} \mathcal{E}_\rho^{(p)}(G) = \inf_{\rho \in A(\Gamma)} \mathcal{E}_\rho^{(p)}(G) = \inf_{\rho \in A^*(\Gamma)} \mathcal{E}_\rho^{(p)}(G)$$

Lemma:

- There exists an extremal density $\rho^* : E \to \mathbb{R}$.
- If $1 < p < \infty$, then ρ^* is unique.

Theorem (Albin, Poggi–Corradini, Sahneh, Goering): There exists an *essential subfamily* $\Gamma^* \subseteq \Gamma$ for which

- Γ^* is finite
- $A(\Gamma^*) = A(\Gamma)$

Theorem (Albin, Poggi–Corradini): There exists a *minimal subfamily* $\tilde{\Gamma} \subseteq \Gamma$ for which

- $\mathrm{Mod}_p(G, \tilde{\Gamma}) = \mathrm{Mod}_p(G, \Gamma)$
- For all $\gamma \in \tilde{\Gamma}$, $\mathrm{Mod}_p(G, \tilde{\Gamma} \setminus \{\gamma\}) < \mathrm{Mod}_p(G, \Gamma)$

Algorithm 1 Approx. $\text{Mod}_p(\Gamma)$ with error tolerance $0 < \epsilon_{\text{tol}} < 1$.

```
ρ ← 0
Γ′ ← ∅
loop
    γ ← shortest(ρ)                          ▷ e.g. Dijkstra's algorithm
    if ℓ_ρ(γ)^p ≥ 1 − ε_tol then
        stop
    end if
    Γ′ ← Γ′ ∪ {γ}
    ρ ← argmin{E_p(ρ) | ρ ∈ A(Γ′)}           ▷ Convex optimization
end loop
```

Theorem: The output Γ' and ρ satisfy

$$\frac{\text{Mod}_p(G,\Gamma) - \text{Mod}_p(G,\Gamma')}{\text{Mod}_p(G,\Gamma)} \le \epsilon_{\text{tol}}, \quad \frac{||\rho^* - \rho||_p}{||\rho^*||_p} \le \begin{cases} 2^{1-1/p}{\epsilon_{\text{tol}}}^{1/p} & p \ge 2 \\ \left(\frac{2}{p-1}\epsilon_{\text{tol}}\right)^{1-1/p} & p < 2 \end{cases}$$

Let

- Γ a family of curves on $G = (V, E, \sigma)$
- $\rho \in A(\Gamma)$

Define

- the ∞-energy $\mathcal{E}_\rho^{(\infty)}(G) := \lim_{p\to\infty} \mathcal{E}_\rho^{(p)}(G)^{\mathbf{1/p}} = \max_{e\in E} |\rho(e)|$
- the ∞-**modulus** $\mathrm{Mod}_\infty(G, \Gamma) := \inf_{\rho\in A(\Gamma)} \mathcal{E}_\rho^{(\infty)}(G)$

Theorem: $\mathrm{Mod}_\infty(G, \Gamma) = \dfrac{1}{\ell(\Gamma)}$, where

- $\ell(\gamma)$ is the (unweighted) length of a walk γ
- $\ell(\Gamma) = \inf_{\gamma\in\Gamma} \ell(\gamma)$

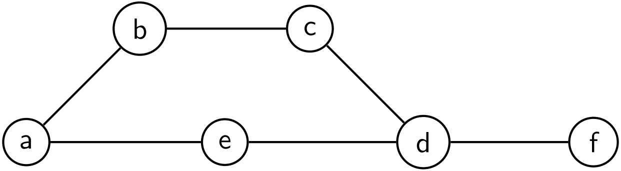
Connecting Family Theorem (3/3): Let

- $G = (V, E, \sigma)$ be undirected
- $\Gamma(s, t)$ be the family of walks from s to t

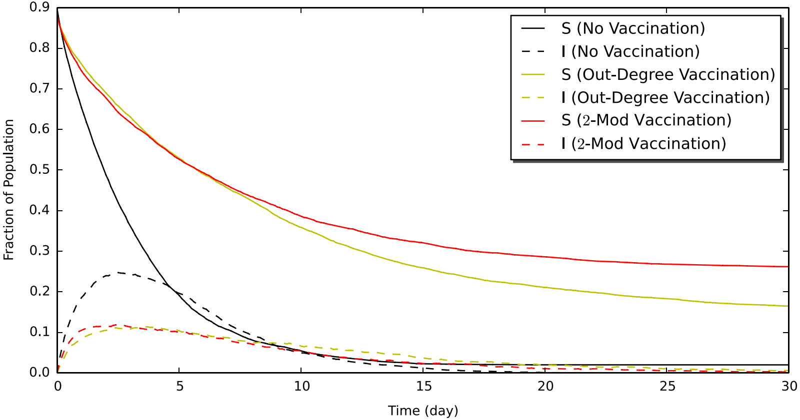
Then

- $\mathrm{Mod}_1(G, \Gamma(s, t)) =$ maximum flow from s to t.
- $\mathrm{Mod}_2(G, \Gamma(s, t)) =$ effective conductance between s and t.
- $\mathrm{Mod}_\infty(G, \Gamma(s, t)) = \dfrac{1}{\text{unweighted distance from } s \text{ to } t.}$

Example:

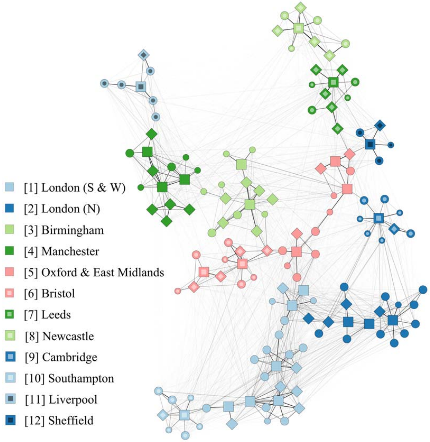


Application: Vaccination strategies based on 2-modulus centrality versus out-degree in random modular networks



J. Comput. Appl. Math. (2016), **307**: 307–318

Prospect: Transfer network constructed from claims data



PLoS ONE (2012) **7**(4): e35002

Interpretation: Expected usage under energy minimization

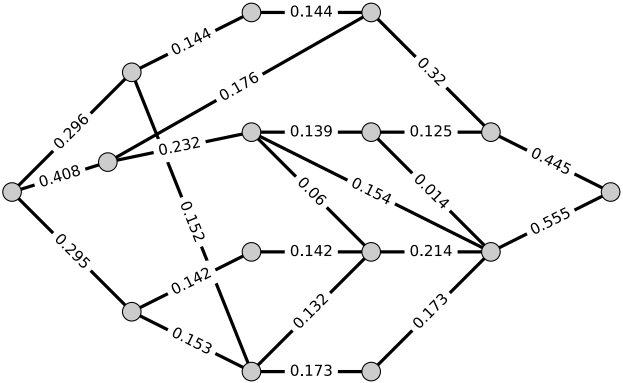
Theorem (Albin, Poggi–Corradini): Let

- $\mathcal{N} = \mathcal{N}(\gamma, e) \in \mathbb{N}^{|\Gamma| \times |E|}$ be the number of steps of γ along e
- $\mu : \Gamma \to [0, 1]$ range over the probability mass functions on Γ

Then

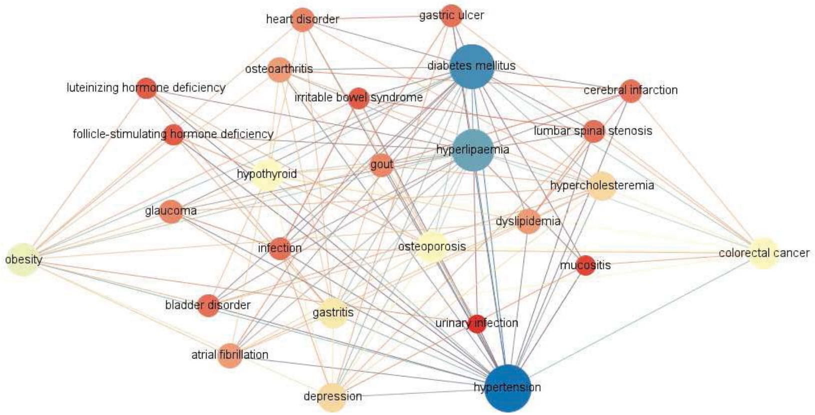
- $\mathrm{Mod}_2(G, \Gamma) = \dfrac{1}{\min_\mu \mu' \mathcal{N} \mathcal{N}' \mu} = \dfrac{1}{\min_\mu \sum_e \mathbb{E}[\mathcal{N}(\underline{\gamma}, e)]^2}$
- For all $e \in E$, $\dfrac{\rho^*(e)}{\mathrm{Mod}_2(G, \Gamma)} = (\mathcal{N}' \mu^*)(e) = \mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)]$

Example: Edges e weighted by $\dfrac{\rho^*(e)}{\mathrm{Mod}_2(G,\Gamma)}$



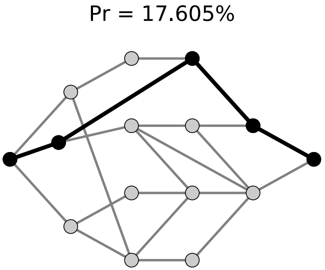
arXiv: 1605.08462

Prospect: Comorbidity network constructed from EHR data



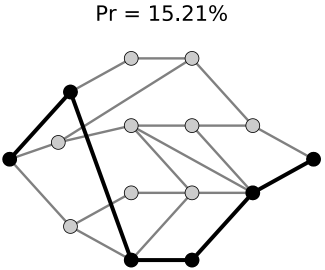
AMIA Jt Summits Transl Sci Proc. (2015): 201–206

Example: Minimal family with energy-minimizing probabilities



arXiv: 1605.08462

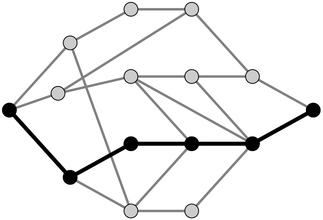
Example: Minimal family with energy-minimizing probabilities



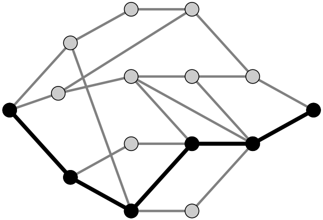
arXiv: 1605.08462

Example: Minimal family with energy-minimizing probabilities

Pr = 12.801%



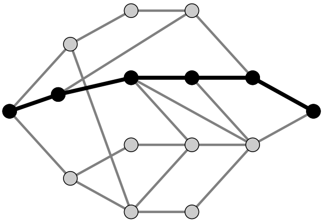
Pr = 8.603%



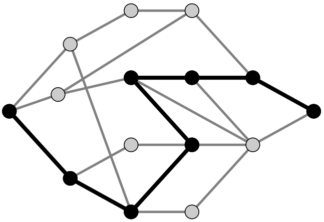
arXiv: 1605.08462

Example: Minimal family with energy-minimizing probabilities

Pr = 7.873%



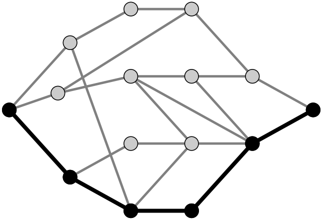
Pr = 4.597%



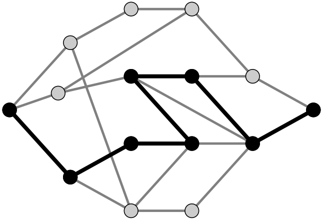
arXiv: 1605.08462

Example: Minimal family with energy-minimizing probabilities

Pr = 2.092%

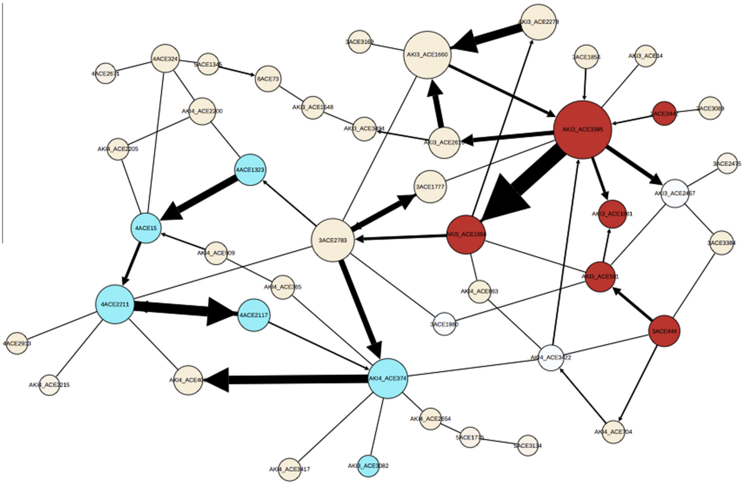


Pr = 1.445%



arXiv: 1605.08462

Prospect: Clinical pathways recovered from administrative logs



J. Biomed. Inform. (2015): **58**: 186–197

Fin.