# Center and Composition conditions for Abel Equation

Thesis for the degree of Doctor of Philosophy

by

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# Abstract

We consider the real vector field $(f(x,y), g(x,y))$ on the real plane $I\!R^2$. This vector field describes the dynamical system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} .$$

A classical Center-Focus problem, which was stated by H. Poincare in 1880's, is: find conditions on $f$, $g$, under which all trajectories of this dynamical system are closed curves around some point. This situation is called **a center**. In some cases this problem is reduced to the problem of finding conditions for solutions of Abel differential equation

$$\rho' = p(\theta)\rho^2 + q(\theta)\rho^3$$

to be periodic on the interval $[0, 2\pi]$.

In the thesis various special cases of the Center-Focus problem for Abel Equation are investigated, through their relation to Composition algebra of polynomials and rational functions, to Generalized Moments and to Algebraic Geometry of Plane Curves.

# Contents

# Chapter 1

# Introduction

Let $F(x,y)$, $G(x,y)$ be algebraic polynomials in $x$, $y$ without free and linear terms. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y) \\ \dot{y} = x + G(x,y) \end{cases} \tag{1.1}$$

One can reduce the system (1.1) with homogeneous $F$, $G$ of degree $d$ to **Abel differential equation**

$$\rho' = p(\theta)\rho^2 + q(\theta)\rho^3 \tag{1.2}$$

where $p(\theta)$, $q(\theta)$ are polynomials in $\sin\theta$, $\cos\theta$ of degrees depending only on $d$. Then (1.1) **has a center** if and only if (1.2) has all the solutions periodic on $[0, 2\pi]$, i.e. solutions $\rho = \rho(\theta)$ satisfying $y(0) = y(2\pi)$.

The classical center problem as stated for (1.2) is to find conditions on $p$ and $q$ such that (1.2) has a center. We shall call it **Center Condition**.

The following simple sufficient condition was introduced in [AL]. Let $w(\theta) \in C^1[0, 2\pi]$ be a function such that $w(0) = w(2\pi)$. Let

$$\begin{cases} p(\theta) = \tilde{p}(w(\theta))w'(\theta) \\ q(\theta) = \tilde{q}(w(\theta))w'(\theta). \end{cases} \tag{1.3}$$

Then all the solutions of (1.2) have the form $\rho(\theta) = \tilde{\rho}(w(\theta))$, hence they satisfy the condition $\rho(0) = \rho(2\pi)$.

We shall call the representation (1.3) **Composition Condition** on $p$, $q$. The composition condition is **sufficient**, but **not necessary** for the center.

The main subject of this research is the investigation of relations between center and composition conditions.

In the next chapter I introduce statement of center problem for vector fields on the plane. Reduction of the problem to the problem for Abel differential equation (Cherkas transform) is demonstrated and an attempt is done to generalize it. Center and composition conditions are investigated for dynamical systems on the plane with $F$, $G$ – homogeneous polynomials of degrees 2 and 3. We show which center components can be represented as a composition and which can not.

In the third chapter I introduce Poincare return map for Abel differential equation. Recurrence relations and corresponding ideals are studied and generators for these ideals are found.

In the fourth chapter of the thesis Abel differential equation is studied with $p$, $q$ – trigonometric polynomials of small degrees (up to 2). It is shown that in these cases center and composition conditions coincide, although it is known that for greater degrees of $p$, $q$ some center conditions are not given by composition.

In the fifth chapter of the thesis center and composition conditions are studied for Abel differential equation with $p$, $q$ – algebraic polynomials of small degrees. It was shown that these conditions coincide. Some generalization of center problem are introduced. We show also some special families of polynomials, for which equivalence of the center and the composition conditions can be easily shown.

In the sixth chapter Riemann Surface approach to the Center problem for Abel equation is discussed. This is a convenient general setting, where Abel Equation is considered on a given Riemann Surface. The notions of center and composition are generalized accordingly.

In the seventh chapter we study composition for rational functions. We establish some facts about structure of composition for algebraic polynomials and Laurent polynomials.

In the eighth chapter center problem is studied for Abel differential equation with rational functions $p$, $q$. One can rewrite the differential equation (1.2) in a complex form, expressing $\sin\theta$ and $\cos\theta$ through $z = e^{i\theta}$, i.e.

$$\begin{cases} \cos\theta = \frac{z+z^{-1}}{2}, \\ \sin\theta = \frac{z-z^{-1}}{2i}, \\ \rho = y, \end{cases}$$

so one obtains $p$ and $q$ in the form of **Laurent polynomials** in $z$, and Abel differential equation is

$$\frac{dy}{dz} = p(z)\, y^2 + q(z)\, y^3$$

considered on the circle $z([0, 2\pi]) = S^1$. It is shown that certain integral condition on $P = \int p$ and $Q = \int q$ (namely vanishing of **generalized moments** $\int_{|z|=1} P^i Q^j dP = 0$) implies center.

Finally, in the last chapter of the thesis Riemann-surface approach is extended for the case of elliptic functions. Specifically, for elliptic functions $P$, $Q$ it is shown that in general the condition $\int_{|z|=1} P^i Q^j dP = 0$ does not imply center. The difference between the Laurent polynomials and elliptic functions $P$, $Q$ is shown to be in the topology of the complex curve $Y = \{(P(z), Q(z)) : z \in \mathbb{C}\} \subseteq \mathbb{C}^2)$. In the first case this curve is rational, and in the second case it is topologically a torus. The one-dimensional homology of the torus is responsible for ramification of the solutions of the Abel equation, in spite of vanishing of all the moments $\int_{|z|=1} P^i Q^j dP$.

# Chapter 2

# Center and Composition Conditions for vector fields on the plane

## 2.1 Introduction

The following formulation of the center problem (see e.g. [Sch] for a general discussion of the classical center problem) is considered: Let $F(x,y)$, $G(x,y)$ be algebraic homogeneous polynomials in $x$, $y$ of degree $d$. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y) \\ \dot{y} = x + G(x,y) \end{cases} \tag{2.1}$$

A solution $x(t)$, $y(t)$ of (2.1) is said to be **closed** if it is defined in the interval $[0,t_0]$ and $x(0) = x(t_0)$, $y(0) = y(t_0)$. The system (2.1) is said to **have a center at** 0 if all the solutions around zero are closed. Then the general problem is: under what conditions on $F,G$ the system (2.1) has a center at zero?

It was shown in [Ch] that one can reduce the system (2.1) with homogeneous $F$, $G$ of degree $d$ to Abel equation

$$\rho' = p(\varphi)\rho^2 + q(\varphi)\rho^3 \tag{2.2}$$

where $p(\varphi)$, $q(\varphi)$ are polynomials in $\sin\varphi$, $\cos\varphi$ of degrees $d+1$ and $2d+2$

respectively. Then (2.1) has a center if and only if (2.2) has all the solutions periodic on $[0, 2\pi]$, i.e. all the solutions $r = r(\varphi)$ with $\rho(0)$ sufficiently small satisfy $\rho(0) = \rho(2\pi)$. Conditions on coefficients of $p$, $q$, under which (2.2) has all the solutions periodic, are called **center conditions for Abel equation** (2.2).

Following [AL], in [BFY1-BFY3] the following simple sufficient condition was stated. Let $P(\varphi) = \int_0^{\varphi} p(s)ds$, $Q(\varphi) = \int_0^{\varphi} q(s)ds$. Then:

**Composition Condition.** *The sufficient condition for the existence of the center for (2.2) is the representability of $P(\varphi)$, $Q(\varphi)$ as a composition of algebraic polynomials with trigonometric polynomial, i.e. there exist a trigonometric polynomial $W(\varphi)$ - polynomial of $\cos\varphi$, $\sin\varphi$ s.t. $W(0) = W(2\pi) = 0$, and algebraic polynomials $\tilde{P}$, $\tilde{Q}$ without free terms, s.t. $P(\varphi) = \tilde{P}(W(\varphi))$, $Q(\varphi) = \tilde{Q}(W(\varphi))$.*

The sufficiency follows from the fact, that if $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, then the solution of (2.2) $\rho(\varphi) = \tilde{\rho}(W)$, and as $W(0) = W(2\pi)$, $\rho(0) = \rho(2\pi)$.

The full center conditions are known yet only for homogeneous perturbations $F$, $G$ of degrees $d = 2, 3$. In this chapter we

- demonstrate Cherkas transform using normal form technique (sections 2.2–2.3);
- show following [AL] that Hamiltonian and Symmetric components are given in fact by composition (section 2.4);
- explicitly write down composition representation for Hamiltonian and Symmetric components in quadratic (section 2.5) and cubic (section 2.6) cases;
- show by counterexample that other known components for quadratic and cubic homogeneous vector fields are in general unrepresentable as a composition.

## 2.2 Cherkas transform

We shall show how in a special case of systems (2.1), the problem of finding center conditions on $F(x, y)$, $G(x, y)$ can be reduced to the center problem for Abel equation (2.2).

Write (2.1) in polar coordinates $x = r\ \cos\theta$, $y = r\ \sin\theta$. We obtain

$$\begin{cases} \dot{x} = \dot{r}\ \cos\theta - r\ \sin\theta \cdot \dot{\theta} = -r\ \sin\theta + F(r\ \cos\theta, r\ \sin\theta) \\ \dot{y} = \dot{r}\ \sin\theta - r\ \cos\theta \cdot \dot{\theta} = r\ \cos\theta + F(r\ \cos\theta, r\ \sin\theta) \end{cases}$$

Multiplying the first equation by $\cos\theta$, the second by $\sin\theta$ and adding, we get

$$\dot{r} = F(r\ \cos\theta, r\ \sin\theta)\cos\theta + G(r\ \cos\theta, r\ \sin\theta)\sin\theta$$
$$= r^d(F(\cos\theta, \sin\theta)\cos\theta + G(\cos\theta, \sin\theta)\sin\theta) = r^d f(\theta)$$

In a similar way,

$$\dot{\theta} = 1 + r^{d-1}(-F(\cos\theta, \sin\theta)\sin\theta + G(\cos\theta, \sin\theta)\cos\theta) = 1 + r^{d-1}g(\theta)$$

Finally, we get

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r^d f(\theta)}{1 + r^{d-1}g(\theta)}$$

Then we apply a transformation, suggested by L. Cherkas in [Che1]:

$$\begin{cases} \rho = \dfrac{r^{d-1}}{1 + r^{d-1}g(\theta)} \\ r^{d-1} = \dfrac{\rho}{1 - \rho g(\theta)} \end{cases}$$

Notice that for $d = 2$ this transformation and its inverse are regular at $r = \rho = 0$. For $d > 2$, Cherkas' transformation, considered over complex numbers, ramifies at zero, but for small real $r > 0$, $\rho > 0$, it is monotone.

Denote $1 + r^{d-1}g(\theta)$ by $A$. We have

$$\frac{d\rho}{d\theta} = \frac{1}{A^2}\left((d-1)r^{d-2} \cdot \frac{dr}{d\theta} \cdot A - r^{d-1}\left((d-1)r^{d-2}\frac{dr}{d\theta}g(\theta) + r^{d-1}g'(\theta)\right)\right)$$

$$= \frac{r^{d-1}}{A^2}\left(\left((d-1)f(\theta) + g'(\theta)\right)r^{d-1} - (d-1)f(\theta)g(\theta)\frac{r^{2d-2}}{A}\right)$$

$$= ((d-1)f(\theta) + g'(\theta))\,\rho^2 - (d-1)f(\theta)g(\theta)\rho^3$$

Thus we obtain

$$\frac{d\rho}{d\theta} = p(\theta)\rho^2 + q(\theta)\rho^3$$

where $p(\theta) = (d-1)f(\theta) + g'(\theta)$ and $q(\theta) = -(d-1)f(\theta)g(\theta)$ are polynomials in $\sin\theta$, $\cos\theta$ of degrees $d+1$ and $2d+2$, respectively.

Clearly, the trajectory of (2.1) near the origin is closed if and only if the corresponding solution of (2.2) satisfies $\rho(2\pi) = \rho(0)$ (and hence is periodic). Therefore the center-focus problem for (2.1) is translated into the problem of finding conditions on $p$ and $q$, under which all the solutions of (2.2) are periodic.

## 2.3 General transform from a plane vector field to a first order non-autonomous differential equation

In this section a transform from a polynomial plane vector field to a first order non-autonomous differential equation is illustrated. Let $F(x, y)$, $G(x, y)$ be arbitrary polynomials of degree grater than 2. We can consider any polynomials $F(x, y)$, G(x,y) as a sum of homogeneous polynomials $F_d$, $G_d$ of degree $d$:

$$\begin{cases} F = F_2 + F_3 + \ldots + F_k \\ G = G_2 + G_3 + \ldots + G_k \end{cases}$$

In a similar way to the homogeneous case we obtain

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r^2 f_2(\theta) + r^3 f_3(\theta) + \ldots + r^k f_k(\theta)}{1 + r g_2(\theta) + r^2 g_3(\theta) + \ldots + r^{k-1} g_k(\theta)}$$

When $r$ is sufficiently small, we can write it as

$$\frac{dr}{d\theta} = \left(r^2 f_2(\theta) + r^3 f_3(\theta) + \ldots + r^k f_k(\theta)\right) \sum_{i=1}^{\infty} (r g_2(\theta) + r^2 g_3(\theta) + \ldots + r^{k-1} g_k(\theta))^i$$

After multiplication we obtain formal infinite power series:

$$\dot{r} = \sum_{i=2}^{\infty} a_i\, r^i,$$

where $a_i$ are polynomials in $\sin\theta$, $\cos\theta$.

Now we can apply normal form technique, eliminating one by one all the terms of degrees higher than 3. Namely, substituting $u = r + \epsilon(t)\, r^4$, we get

$$\dot{u} = \dot{r} + \dot{\epsilon}\, r^4 + 4\epsilon\, r^3 \dot{r} = \dot{r}\,(1+4\epsilon\, r^3) + \dot{\epsilon}\, r^4 = (\sum_{i=2}^{\infty} a_i\, r^i)(1+4\epsilon\,(\sum_{i=2}^{\infty} a_i\, r^i)^3) + \dot{\epsilon}\,(u - \epsilon\, u^4)^4$$

$$= a_2 u^2 + a_3 u^3 + (a_4 + \epsilon) u^4 + \ldots.$$

So by choosing $\dot{\epsilon} = -a_4$ we get differential equation with the right hand side – power series without $u^4$. Continuing elimination, we can kill all terms $u^k$ for $k \geq 0$ (there are some obstructions to this process, which we don't discuss now). These initial computations suggest a way to transform any polynomial vector field to a first order non-autonomous differential equation.

## 2.4 Composition conditions for Symmetric and Hamiltonian components in general case

### 2.4.1 Hamiltonian system

We consider the system

$$\begin{cases} \dot{x} = -y - \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = x + \frac{\partial H}{\partial x}(x, y) \end{cases}$$

with $H(x, y)$ – homogeneous polynomial in $x$, $y$ of degree $d+1$. After transforming to polar coordinates we get

$$\begin{cases} \dot{r} = r^d f(\theta) \\ \dot{\theta} = 1 + r^{d-1} g(\theta) \end{cases}$$

with

$$f(\theta) = -\frac{\partial H}{\partial y}(\cos\theta \sin\theta)\cos\theta + \frac{\partial H}{\partial x}(\cos\theta, \sin\theta)\sin\theta,$$

hence $f(\theta) = -\frac{dH}{d\theta} H(\cos\theta, \sin\theta)$.

$$g(\theta) = \frac{\partial H}{\partial y}(\cos\theta \sin\theta)\sin\theta + \frac{\partial H}{\partial x}(\cos\theta, \sin\theta)\cos\theta.$$

Due to homogeneity of $H$ $g(\theta) = (d+1)H(\cos\theta, \sin\theta)$.

Applying Cherkas transformation, we obtain the Abel equation $\rho' = p(\theta)\rho^2 + q(\theta)\rho^3$, with $p(\theta) = (d-1)f(\theta) - g'(\theta)$, $q(\theta) = -(d-1)f(\theta)g(\theta)$. Integrating, we get

$$P(\theta) = -(d-1)H(\cos\theta, \sin\theta) - (d+1)H(\cos\theta, \sin\theta) + C_1 = -2dH(\cos\theta, \sin\theta) + C_1.$$

$$q(\theta) = (d^2-1)\frac{dH(\cos\theta, \sin\theta)}{dt}H(\cos\theta, \sin\theta),$$

hence

$$Q(\theta) = \frac{d^2-1}{2}H^2(\cos\theta, \sin\theta) + C_2,$$

and

$$Q(\theta) = \frac{d^2-1}{8d^2}P^2(\theta) + C_3 P(\theta),$$

where $C_3$ is a constant.

### 2.4.2 Symmetric component

We consider the planar system

$$\begin{cases} \dot{x} = -y + F(x, y) \\ \dot{y} = x + G(x, y) \end{cases}$$

As we can rotate our system around the origin, it is enough to consider the case of symmetry with respect to $x$-axis:

$$\begin{cases} F(x, -y) = -F(x, y) \\ G(x, -y) = G(x, y) \end{cases}$$

In polar coordinates we get $f(-\theta) = F(\cos\theta, -\sin\theta)\cos\theta + G(\cos\theta, -\sin\theta)(-\sin\theta) = -f(\theta)$, $g(-\theta) = -F(\cos\theta, -\sin\theta)(-\sin\theta) + G(\cos\theta, -\sin\theta)\cos\theta = g(\theta)$.

After Cherkas transformations we get $p(\theta) = (d-1)f(\theta) - g'(\theta)$, $q(\theta) = -(d-1)f(\theta)g(\theta)$, hence $p(\theta)$ and $q(\theta)$ are odd functions, hence $P(\theta)$ and $Q(\theta)$ are even functions, i.e. they are functions of $\cos\theta$.

Below we write down explicit expressions for coefficients of such compositions for Hamiltonian and Symmetric cases for homogeneous polynomial planar systems of the second and the third degrees.

## 2.5 Composition and center conditions for quadratic planar systems

The system (2.1) with homogeneous $F(x,y)$, $G(x,y)$ of degree 2 can be rewritten in the complex notations ( Zoladek [Zo1]), with 6 parameters:

$$\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2,$$

where $z = x + iy$, $A = a_1 + ia_2$, $B = b_1 + ib_2$, $C = c_1 + ic_2$.

It is known (see [Zo1]), that the components of the center set in this case are

$$B = 0; \qquad \text{Lotka-Volterra component } (Q_3^{LV})$$
$$Im(AB) = Im(\bar{B}^3 C) = Im(A^3 C) = 0; \qquad \text{Symmetric component } (Q_3^R)$$
$$2A + \bar{B} = 0; \qquad \text{Hamiltonian component } (Q_3^H)$$
$$A - 2\bar{B} = |C| - |B| = 0. \qquad \text{Darbu component } (Q_4)$$

In usual notations on the real plane we get the following system:

$$\begin{aligned} \dot{x} &= +y + (a_1 + b_1 + c_1)x^2 + (-2a_2 + 2c_2)xy + (-a_1 + b_1 - c_1)y^2 \\ \dot{y} &= -x + (a_2 + b_2 + c_2)x^2 + (+2a_1 - 2c_1)xy + (-a_2 + b_2 - c_2)y^2 \end{aligned}$$

The center conditions are as follow:

$$b_1 = 0, \; b_2 = 0, \qquad (Q_3^{LV})$$

$$\begin{cases} a_1 b_2 + a_2 b_1 = 0 \\ b_2^3 c_1 + b_1^3 c_2 - 3b_1 b_2^2 c_2 - 3b_1^2 b_2 c_1 = 0 \\ -a_2^3 c_1 + a_1^3 c_2 - 3a_1 a_2^2 c_2 + 3a_1^2 a_2 c_1 = 0 \end{cases}, \qquad (Q_3^R)$$

$$2a_1 + b_1 = 0, \; 2a_2 - b_2 = 0, \qquad (Q_3^H)$$

$$\begin{cases} a_1 - 2b_1 = 0 \\ a_2 + 2b_2 = 0 \\ c_1^2 + c_2^2 = b_1^2 + b_2^2 \end{cases} . \tag{$Q_4$}$$

We shall find the direct representations of Symmetric and Hamiltonian components as a composition, and we shall show that in general case Lotka-Volterra and Darbu components can not be represented as a composition.

### 2.5.1 The composition condition for the Hamiltonian component

**Lemma.** *In Hamiltonian case the system has the form*

$$\begin{aligned} \dot{x} &= +y + (-a_1 + c_1)x^2 + (-2a_2 + 2c_2)xy + (-3a_1 - c_1)y^2 \\ \dot{y} &= -x + (3a_2 + c_2)x^2 + (+2a_1 - 2c_1)xy + (a_2 - c_2)y^2 \end{aligned}$$

*and there exists a composition of the form*

$$Q(x) = \frac{3}{32}(P^2(x) + \frac{8}{3}(3a_2 + c_2)P(x))$$

**Proof:** Using the *Mathematica* program (here is the main function):

```
F2[x_,y_]:=-lambda3 x^2  +lambda6 y^2;
G2[x_,y_]:= +(2 lambda3 + lambda4) x y ;
f=G2[Cos[fi],Sin[fi]]*Sin[fi]+F2[Cos[fi],Sin[fi]]*Cos[fi];
g=G2[Cos[fi],Sin[fi]]*Cos[fi]-F2[Cos[fi],Sin[fi]]*Sin[fi];
pa=TrigExpand[f-D[g,fi]];
qa=TrigExpand[-f*g];
P=TrigReduce[Integrate[pa,{fi,0,x}]];
Q=TrigReduce[Integrate[qa,{fi,0,x}]];
```

we find

$P = -(4/3)(-3a_2 - c_2 + 3a_2\cos(x) + c_2\cos(3x) + 3a_1\sin(x) - c_1\sin(3x)),$

$Q = (1/12)(9a_1^2 - 9a_2^2 + c_1^2 - 12a_2c_2 - c_2^2 - 9a_1^2\cos(2x) + 9a_2^2\cos(2x) - 6a_1c_1\cos(2x) + 6a_2c_2\cos(2x) + 6a_1c_1\cos(4x) + 6a_2c_2\cos(4x) - c_1^2\cos(6x) +$

$c_2^2\cos(6x) + 18a_1a_2\sin(2x) - 6a_2c_1\sin(2x) - 6a_1c_2\sin(2x) - 6a_2c_1\sin(4x) + 6a_1c_2\sin(4x) - 2c_1c_2\sin(6x))$.

Now to choose $\lambda$ in the representation $Q(x) = \frac{3}{32}(P^2(x) + \lambda P(x))$ we compute $P(\pi) = (8/3)(3a_2 + c_2)$, $Q(\pi) = 0$, hence $\lambda = -(8/3)(3a_2 + c_2)$, and the identity

$$Q(x) = \frac{3}{32}(P^2(x) + \frac{8}{3}(3a_2 + c_2)P(x))$$

holds. ■

### 2.5.2 Composition condition for the symmetric component

**Lemma.** *In Symmetric case there exists a composition with the polynomial of the first degree of the form* $W(x) = \cos x + \alpha\sin x - 1$, *where* $\alpha$ *is a constant which is found from the cubic equation involving coefficients of the system.*

**Proof:** In this case the center conditions are

$$\begin{cases} a_1b_2 + a_2b_1 = 0 \\ b_2^3c_1 + b_1^3c_2 - 3b_1b_2^2c_2 - 3b_1^2b_2c_1 = 0 \\ -a_2^3c_1 + a_1^3c_2 - 3a_1a_2^2c_2 + 3a_1^2a_2c_1 = 0 \end{cases}.$$

Without loss of generality we may assume that $b_2 \neq 0$ (if both $b_1$ and $b_2$ are zeroes, we are in Lotka-Volterra case). Then from the first equation

$$a_1 = -a_2b_1/b_2,$$

and from the two last equation we can find: either

$$3b_1^2 = b_2^2, \quad c_2 = 0$$

or

$$3b_1^2 \neq b_2^2, \quad c_1 = \frac{1}{3}b_1c_2(\frac{1}{b_2} - \frac{8b_2}{3b_1^2 - b_2^2}).$$

Consider the first case, when $b_2 = b_1\sqrt{3}$. Running *Mathematica*, we get

$$P = 2\sqrt{3}b_1 - 2\sqrt{3}b_1\cos(x) + 2b_1\sin(x) + 4/3c_1\sin(3x),$$

$Q = (1/12)(2a_2^2 - 6b_1^2 - 2\sqrt{3}a_2c_1 + 3b_1c_1 + c_1^2 - 2a_2^2\cos(2x) + 6b_1^2\cos(2x) + 2\sqrt{3}a_2c_1\cos(2x) - 3b_1c_1\cos(4x) - c_1^2\cos(6x) + 2\sqrt{3}a_2^2\sin(2x) - 6\sqrt{3}b_1^2\sin(2x) - 6a_2c_1\sin(2x) - 3\sqrt{3}b_1c_1\sin(4x))$.

After some computations we obtain composition with the first degree polynomial $W(x) = \cos(x) - \dfrac{1}{\sqrt{3}}\sin(x) - 1$:

$P(x) = (-2c_1\sqrt{3})W(x)^3 - (6c_1\sqrt{3})W(x)^2 - (4c_1\sqrt{3} + 2b_1\sqrt{3})W(x)$,

$$Q = (\frac{9}{8}c_1^2)W^6 + \frac{27c_1^2}{4}W^5 + \frac{117c_1^2 + 18b_1c_1}{8}W^4 + \frac{18b_1c_1 + 27c_1^2}{2}W^3$$
$$+\frac{-a_2^2 + 3b_1^2 + a_2c_1\sqrt{3} + 21b_1c_1 + 9c1^2}{2}W^2 + (-a_2^2 + 3b_1^2 + \sqrt{3}a_2c_1 + 3b_1c_1)W.$$

Other two cases were considered similarly, and in each of them we got composition. ■

### 2.5.3 Non-composition components for quadratic systems

One of ways to check if two polynomials $P(x)$, $Q(x)$ have a common divisor is to plot graphs of polynomials $P(x) - P(y)$ and $Q(x) - Q(y)$ on the real plane $(x, y)$ and to see if there is a common subgraph outside the diagonal $x = y$. For real curves these subgraphs are real. (see [ER] for detail).

We plot graphs for different numerical values in each of the four components , using the "Matlab" software.

For Lotka-Volterra component we consider the numerical example with parameters $a_1 = 1$, $a_2 = 3$, $c_1 = 2$, $c_2 = 3$, i.e.

$$\begin{cases} F(x,y) = 3x^2 - 3y^2 \\ G(x,y) = 6x^2 - 2xy - 6y^2 \end{cases}.$$

For Darbu component we consider the numerical example with parameters $a_1 = 1$, $a_2 = 3$, $c_1 = 2$, $c_2 = 3$, i.e.

$$\begin{cases} F(x,y) = 3x^2 - 3y^2 \\ G(x,y) = 6x^2 - 2xy - 6y^2 \end{cases}.$$

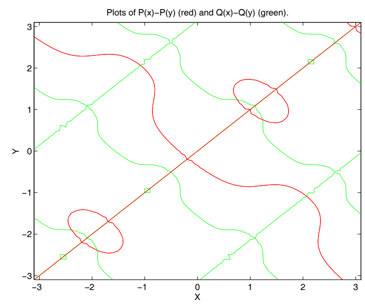



Figure 2.1: Graph for Lotka-Volterra component

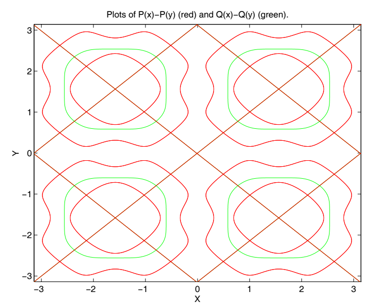



Figure 2.2: Graph for Darbu component

For Lotka-Voltera and Darbu components we see that there is no common subgraph. Although rigorous proof would require discussion of computer precision of Matlab software and graphical capabilities of hardware, we assume that visually observable distance between curves is greater than machine error, and there is no common subgraph. These graphical counterexamples prove that there is no composition for these two components.

## 2.6 Cubic polynomial vector fields

The system (2.1) with homogeneous $F(x,y)$, $G(x,y)$ of degree 3 can be rewritten in the complex notations ( Zoladek [Zo2]) with 8 parameters:

$$\dot{z} = iz + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3,$$

where $z = x + iy$, $D = d_1 + id_2$, $E = e_1 + ie_2$, $F = f_1 + if_2$, $G = g_1 + ig_2$.

It is known (see [Zo2]), that the components of the center set in this case are

$$Re(E) = 3D + \bar{F} = 0; \qquad (C_4^H)$$

$$Re(E) = Im(DF) = Re(D^2G) = Re(\bar{F}^2G) = 0; \qquad (C_4^R)$$

$$E = D - 3\bar{F} = |G| - 2|F| = 0. \qquad (C_6)$$

In usual notations on $\mathbb{R}^2$ we get the following system:

$$\begin{aligned}
\dot{x} &= -y + (d_1 + e_1 + f_1 + g_1)x^3 + (-3d_2 - e_2 + f_2 + 3g_2)x^2y \\
&\quad +(e_1 - 3d_1 + f_1 - 3g_1)xy^2 + (d_2 - e_2 + f_2 - g_2)y^3 \\
\dot{y} &= x + (d_2 + e_2 + f_2 + g_2)x^3 + (3d_1 + e_1 - f_1 - 3g_1)x^2y \\
&\quad +(e_2 - 3d_2 + f_2 - 3g_2)xy^2 + (-d_1 + e_1 - f_1 + g_1)y^3
\end{aligned}$$

The center conditions are as follow:

$$e_1 = 0,\ 3d_1 + f_1 = 0,\ 3d_2 - f_2 = 0 \qquad (C_4^H)$$

$$\begin{cases} e_1 = 0,\ d_1f_2 + d_2f_1 = 0, \\ d_1^2g_1 - d_2^2g_1 - 2d_1d_2g_2 = 0, \\ f_1^2g_1 - f_2^2g_1 + 2f_1f_2g_2 = 0, \end{cases} \qquad (C_4^R)$$

$$\begin{cases} e_1 = 0,\ e_2 = 0, \\ d_1 - 3f_1 = 0,\ d_2 + 3f_2 = 0, \\ g_1^2 + g_2^2 = 4(f_1^2 + f_2^2) \end{cases} \qquad (C_6)$$

We shall find the direct representations of Symmetric and Hamiltonian components as a composition, and we shall show that in general case Darbu component can not be represented as a composition.

### 2.6.1 Composition conditions for the Hamiltonian component $(C_4^H)$

**Lemma.** *In Hamiltonian case the system has the form*

$$\begin{aligned}
\dot{x} &= -y + (-2d_1 + g_1)x^3 + (-e_2 + 3g_2)x^2y \\
&\quad +(-6d_1 - 3g_1)xy^2 + (4d_2 - e_2 - g_2)y^3 \\
\dot{y} &= x + (4d_2 + e_2 + g_2)x^3 + (6d_1 - 3g_1)x^2y \\
&\quad +(e_2 - 3g_2)xy^2 + (2d_1 + g_1)y^3
\end{aligned}$$

*and there exists a composition of the form*

$$Q(\varphi) = \frac{1}{9}(P^2(\varphi) - 3(4d_2 + e_2 + g_2)P(\varphi)).$$

**Proof:** After running *Mathematica* we get

$P = (6d_2 + (3g_2)/2 - 6d_2\cos(2x) - 3/2g_2\cos(4x) - 6d_1\sin(2x) + 3/2g_1\sin(4x))$,

$Q = (1/8)(16d_1^2 - 16d_2^2 - 16d_2e_2 + g_1^2 - 16d_2g_2 - 4e_2g_2 - g_2^2) + (1/8)(16d_2e_2\cos(2x) - 8d_1g_1\cos(2x) + 8d_2g_2\cos(2x) - 16d_1^2\cos(4x) + 16d_2^2\cos(4x) + 4e_2g_2\cos(4x) + 8d_1g_1\cos(6x) + 8d_2g_2\cos(6x) - g_1^2\cos(8x) + g_2^2\cos(8x) + 16d_1e_2\sin(2x) - 8d_2g_1\sin(2x) - 8d_1g_2\sin(2x) + 32d_1d_2\sin(4x) - 4e_2g_1\sin(4x) - 8d_2g_1\sin(6x) + 8d_1g_2\sin(6x) - 2g_1g_2\sin(8x))$.

Substituting $x = \pi/2$ we get $P = 12d_2$, $Q = -4d_2(e_2 + g_2)$, hence $\mu = -3(4d_2 + e_2 + g_2)$. Now we get

$$Q(\varphi) \equiv \frac{1}{9}(P^2(\varphi) - 3(4d_2 + e_2 + g_2)P(\varphi)),$$

q.e.d.

So, in Hamiltonian case both $P$ and $Q$ can be expressed as a composition of some usual polynomials with a certain trigonometric polynomial, namely $P$. ■

### 2.6.2 Composition condition for the component $(C_6)$ does not hold

In usual notations on $\mathbb{R}^2$ this component is given by the following system:

$$\begin{aligned} \dot{x} &= -y + (4f_1 + g_1)x^3 + (10f_2 + 3g_2)x^2y \\ &\quad + (-8f_1 - 3g_1)xy^2 + (-2f_2 - g_2)y^3 \\ \dot{y} &= x + (-2f_2 + g_2)x^3 + (8f_1 - 3g_1)x^2y \\ &\quad + (10f_2 - 3g_2)xy^2 + (-4f_1 + g_1)y^3 \end{aligned}$$

with an additional relation on coefficients: $g_1^2 + g_2^2 = 4(f_1^2 + f_2^2))$.

We compute, using program, which utilizes only the first four of the 5 equations. We get
$P = 2f_2 + (3g_2)/2 - 2f_2\cos(2x) - 3/2g_2\cos(4x) + 2f_1\sin(2x) + 3/2g_1\sin(4x))$,

$Q = (1/24)(-48f_1^2 + 48f_2^2 + 80f_1g_1 + 3g_1^2 + 64f_2g_2 - 3g_2^2) + (1/24)(-72f_1g_1\cos(2x) - 72f_2g_2\cos(2x) + 48f_1^2\cos(4x) - 48f_2^2\cos(4x) - 8f_1g_1\cos(6x) + 8f_2g_2\cos(6x) - 3g_1^2\cos(8x) + 3g_2^2\cos(8x) + 72f_2g_1\sin(2x) - 72f_1g_2\sin(2x) + 96f_1f_2\sin(4x) - 8f_2g_1\sin(6x) - 8f_1g_2\sin(6x) - 6g_1g_2\sin(8x))$,

where there is one additional relation on coefficients:

$$g_1^2 + g_2^2 = 4(f_1^2 + f_2^2).$$

After trying to get a composition with polynomials of the first $(\alpha\sin x + \beta\cos - \beta)$, second $(\alpha_1\sin x + \beta_1\cos x + \alpha_2\sin 2x + \beta_2\cos 2x - \beta_1 - \beta_2)$ and the forth (simply $\alpha P$) degrees, we got contradiction.

### 2.6.3 Composition conditions for the Symmetric component $(C_4^R)$

**Lemma.** *In symmetric case there exists a composition either with a polynomial of the first order in* $\sin x$, $\cos x$, *or of the second order. Namely,*
*if* $d_1 \neq 0$ *or* $d_1 = d_2 = 0$, *there exists a composition with polynomial* $W(x) = \cos x + \alpha\sin x - 1$, *where* $\alpha$ *is defined from a cubic equation on* $\alpha$*;*
*if* $d_1 = 0$ *and* $d_2 \neq 0$, *there exists a composition with* $W = \cos 2x$.

**Proof:** Here we have the only one "usable" equation $e_1 = 0$ and three nonlinear equations. But we can solve the system of center conditions, considering separately cases $d_1 = 0$ and $d_1 \neq 0$.

First we use the only condition $e_1 = 0$ to compute $P$ and $Q$. Then we consider several cases.

**1.** Assume first $d_1 \neq 0$. Then we can substitute

$$f_2 = -\frac{d_2 f_1}{d_1}$$

into the third equation, and we get instead of the third equation the following one:

$$\frac{f_1^2}{d_1^2}(d_1^2 g_1 - d_2^2 g_1 - 2d_1 d_2 g_2) = 0.$$

We see, that the third equation disappeared, and we are left with conditions:

$$\begin{cases} e_1 = 0 \\ d_1 \neq 0 \\ d_1^2 g_1 - d_2^2 g_1 - 2d_1 d_2 g_2 = 0. \end{cases}$$

Now we again consider two cases:

**1a).** $d_2 = 0 \quad \Rightarrow \quad g_1 = 0,\ f_2 = 0.$
In that case we found the composition with

$$W(x) = -1 + \cos(x) + \sin(x)$$

$$P = 3g_2 W^4 + 12g_2 W^3 + (2f_1 + 12g_2)W^2 + 4f_1 W$$

$$Q = g_2^2 W^8 + 8g_2^2 W^7 + (\frac{4}{3}f_1 g_2 + 24g_2^2)W^6 + (8f_1 g_2 + 32g_2^2)W^5 +$$

$$+\frac{1}{6}(90g_2^2 + 96f_1 g_2 - 6e_2 g_2 + 3f_1^2 - 3d_1^2)W^4 + \frac{1}{3}(32f_1 g_2 - 12g_2^2 - 12e_2 g_2 + 6f_1^2 - 6d_1^2)W^3 +$$

$$+(-4g_2^2 - f_1 g_2 - 4e_2 g_2 - d_1 g_2 + 2f_1^2 - e_2 f_1 - d_1 e_2 - 2d_1^2)W^2 + (-2f_1 g_2 - 2d_1 g_2 - 2e_2 f_1 - 2d_1 e_2)W$$

**1b).** $d_2 \neq 0$, then

$$g_2 = g_1 \frac{d_1^2 - d_2^2}{2d_1 d_2}$$

In this case we obtain the composition with $W = \cos(x) + \alpha \sin(x) - 1$, where $\alpha$ is found from the cubic equation.

**2)** The second case to consider is $d_1 = 0$. Then we get the system

$$\begin{cases} d_1 = 0,\ d_2 f_1 = 0,\ d_2 g_1 = 0, \\ f_1^2 g_1 - f_2^2 g_1 + 2f_1 f_2 g_2 = 0. \end{cases}$$

Again there are two cases:

**2a)**

$$\begin{cases} d_1 = 0, \ d_2 = 0, \\ f_1^2 g_1 - f_2^2 g_1 + 2 f_1 f_2 g_2 = 0. \end{cases}$$

In this case we obtain a composition with $W = \cos(x) + \alpha \sin(x) - 1$, where $\alpha$ is found from the cubic equation.

**2b)** $d_1 = 0, \ f_1 = 0, \ g_1 = 0.$
$P$ and $Q$ are both functions of $\cos(2x)$. ■

# Chapter 3

# Poincare return map for Abel differential equation and computations of center conditions

## 3.1 Introduction

We consider Abel equation

$$y' = p(x)\, y^2 + q(x)\, y^3 \tag{3.1}$$

with $p$, $q$ – any analytic functions. We can fix any pont $a$ instead of $2\pi$ and ask the same question: Under which conditions on $p$ and $q$ are all trajectories "periodic", i.e. any trajectory starting at 0 with the value $y_0$ assumes the same value $y_0$ at $a$.

In this chapter

- as in [BFY1], we write down recurrence relations for coefficients of Poincare return map;
- using these recurrence relations, we compute several first coefficients manually and find out generators of coefficient ideals;
- we describe an algorithm to find these generators using "Mathematica" symbolic computer system.

## 3.2 Poincare return map for Abel equation and recurrence relations

We may look for solutions of (3.1) in the form of Poincare return map

$$y(x, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(x, \lambda) y_0^k, \tag{3.2}$$

where $y(0, y_0) = y_0$, $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the (finite) set of the coefficients of $p$, $q$. Shortly we will write $v_k(x)$.

Then $y(a) = y(a, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(a) y_0^k$ and hence the condition $y(a) \equiv y(0)$ is equivalent to $v_k(a) = 0$ for $k = 2, 3, \ldots \infty$

One can easily show (by substitution of the expansion (3.2) into the equation (3.1)) that $v_k(x)$ satisfy recurrence relations

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ v_n(0) = 0 \quad \text{and} \\ v_n'(x) = p(x) \sum_{i+j=n} v_i(x) v_j(x) + q(x) \sum_{i+j+k=n} v_i(x) v_j(x) v_k(x), \ n \geq 2 \end{cases} \tag{3.3}$$

It was shown in [BFY1] that in fact the recurrence relations (3.3) can be linearized, i.e. the same ideals $I_k = \{v_1, v_2, \ldots, v_k\}$'s are generated by $\{\psi_1(x), \ldots \psi_k(x)\}$, where $\psi_k(x)$ satisfy linear recurrence relations

$$\begin{cases} \psi_0(x) \equiv 0 \\ \psi_1(x) \equiv 1 \\ \psi_n(0) = 0 \quad \text{and} \\ \psi_n'(x) = -(n-1)\psi_{n-1}(x) p(x) - (n-2)\psi_{n-2}(x) q(x), \ n \geq 2 \end{cases} \tag{3.4}$$

which are much more convenient then (3.3).

Direct computations (including several integrations by part) give the following expressions for the first polynomials $\psi_k(x)$, solving the recurrence relation (3.4) (Here we denote $P(x) = \int_0^x p(t)dt$, $Q(x) = \int_0^x q(t)dt)$ :

$$
\begin{aligned}
\psi_2(x) &= -P(x) \\
\psi_3(x) &= P^2(x) - Q(x) \\
\psi_4(x) &= -P^3(x) + 3P(x)Q(x) - \int_0^x q(t)P(t)dt \\
\psi_5(x) &= P^4(x) - 6P^2(x)Q(x) - \int_0^x q(t)P^2(t)dt \\
&\quad +4P(x)\int_0^x q(t)P(t)dt + \frac{3}{2}Q^2(x) \\
\psi_6(x) &= -P^5(x) + 10P^3(x)Q(x) + 5P(x)\int_0^x q(t)P^2(t)dt \\
&\quad -8Q^2(x)P(x) - 10P^2(x)\int_0^x q(t)P(t)dt + 4Q(x)\int_0^x q(t)P(t)dt \\
&\quad -\int_0^x q(t)P^3(t)dt + \frac{1}{2}\int_0^x p(t)Q^2(t)dt \\
\psi_7(\varphi) &= P^6(\varphi) - 15P^4(\varphi)Q(\varphi) - \int_0^\varphi P^4(t)q(t)dt \\
&\quad +(5Q(\varphi) - 15P^2(\varphi))\int_0^\varphi q(t)P^2(t)dt + (20P^3(\varphi) \\
&\quad -24P(\varphi)Q(\varphi))\int_0^\varphi P(t)q(t)dt + 6P(\varphi)\int_0^\varphi q(t)P^3(t)dt \\
&\quad -3P(\varphi)\int_0^\varphi Q^2(t)p(t)dt + 2(\int_0^\varphi P(t)q(t)dt)^2 \\
&\quad -\frac{5}{2}Q^3(\varphi) + \frac{49}{2}P^2(\varphi)Q^2(\varphi) + 3\int_0^\varphi P(t)p(t)Q^2(t)dt
\end{aligned}
$$

and similar much more longer expression was obtained for $\psi_8(x)$.

## 3.3 Generators of ideals $I_k$

Consequently, we get the following set of generators for the ideals $\tilde{I}_k$, $k = 2, \ldots, 8$,

$$
\begin{aligned}
I_2 &= \{P\} \\
I_3 &= \{P, Q\} \\
I_4 &= \{P, Q, \int qP\} \\
I_5 &= \{P, Q, \int qP, \int qP^2\} \\
I_6 &= \{P, Q, \int qP, \int qP^2, \int (qP^3 - \frac{1}{2}pQ^2)\} \\
I_7 &= \{P, Q, \int qP, \int qP^2, \int (qP^3 - \frac{1}{2}pQ^2), \int (qP^4 - 2PpQ^2)\} \\
I_8 &= \{P, Q, \int qP, \int qP^2, \int (qP^3 - \frac{1}{2}pQ^2), \int (qP^4 - 3PpQ^2), \\
& \int (qP^5 - \frac{1}{2}Q^3p - 23P^3Qq - 77P^2q \int Pq)\}
\end{aligned}
$$

These generators were first computed by Alwash and Lloyd in [AL], using the nonlinear recurrence relation (3.3), and recomputed now using the recurrence relation (3.4).

The problem to find generators of ideals $I_k$ was studied also from combinatorial point of view by J. Devlin ([Dev1], [Dev2]). We tried to write a program to find all the generators, using symbolic computations with "Mathematica" software.

The difficulty lies in that the general formula for $\psi_k(\varphi)$ cannot be deduced. Respectively, we can compute $\psi_k(\varphi)$ using the computer, for each given $p(\varphi)$, $q(\varphi)$, but we cannot compute the formula for a general symbolic form of $p(\varphi)$ and $q(\varphi)$, except for the formula with $(k-1)$ integrations for the $k$-th term. But using integration by parts we can reduce the complexity (the number of integrals) in the general formula, as demonstrated in section 3.2. We attempted to write a program which could compute integrals in a symbolic form using integration by parts. This work was done together with Oleg Lavrovsky, working on this summer project under my supervision.

Mathematica is an integrated technical computing system that allows for numerical and symbolic computations of high complexity. However, there are no libraries that would facilitate manipulation of symbolic functions due to the following two difficulties:
a) $p(\varphi)$, $q(\varphi)$ are symbolic functions, making the use of standard `Integrate[]` and `Differentiate[]` functions of Mathematica inapplicable;
b) an interface is necessary for allowing the user to specify the transformation or exclusion of components during the iteration, because there is a certain reasonable limit on the complexity of transformations that we define. Unending switching loops often rise between two or three possible formats of expressions as the computer is unable to decide on a result above $\psi_3(\varphi)$.

To overcome the first difficulty, we created a new set of functions. The function `i[ ]` is used to represent an integral (replacing the standard function `Integrate[]`) and `dv[]`, correspondingly, the default function `Differentiate[]`.

We compute using recursive function

```
f[n_]:=f[n]=(1-n)i[Expand[p f[n-1]]]-(n-2)i[Expand[q f[n-2]]]
```

with initial conditions `f[0] = 0 , f[1] = 1`.

We start with the functions $p$ and $q$ as arbitrary symbols, we introduce the complimentary set $P$ and $Q$ so the output `f[n]` will contain only $p$, $q$, $P$, $Q$ and integration operation `i[ ]`.

```
dv[P] := p
dv[Q] := q

i[p] := P
i[q] := Q
```

Within the Mathematica "kernel" expressions of `f[n]` are traced back until the first defined values (`f[0]` and `f[1]`) and each successive expression calculated from therein and stored.

The main set of transformations was defined in this fashion:

```
i[x_ + y_] := i[x] + i[y],
dv[x_ + y_] := dv[x] + dv[y],
dv[x_ y_] := dv[x] y + dv[y] x,
i[dv[ x_]]:=x,  dv[i[x_]]:=x,
i[ p P^(n_)]:=(1/(n+1)) P^(n+1)
```

etc

Using these rules, if Mathematica encounters an expression such as

```
 ... i[p + q + i[p P^2]]...,
```

it transforms it in the following way:

... i[p] + i[q + i[p P*P]]...⇒...i[p] + i[q] + i[p P*P]...⇒ ... P+Q+ (1/3) P*P*P...

This continues until the rule is no longer applicable. To overcome the second difficulty we define integration by parts as

```
i[p B_] := P B - i[P [dv[B]]] /; Test;
```

where $B$ is any expression. Test represents a procedure to display a prompt for accepting or denying transforming of the given expression. Note that it was critical that the ordering of these expressions for execution was carefully determined during analysis of such problems, so that the passing of conversions should follow effective sequences. As we can apply many different integration by parts (taking different expressions as derivatives under integration sign), this step requires some complicated testing procedures.

A set of functions was also added that allow the user to choose terms to break down further, again progressing the formula into a more preferable form. This was based on separately transforming the chosen terms and inserting the result into the general expression, then simplifying.

The developed algorithm proved to be successful in the evaluation of $\psi_k(\varphi)$ in the extent of comparison with currently known expressions (up to $k = 7$), and had the required integral complexity in higher expressions. For comparison, the manual computation of $\psi_8(\varphi)$ takes many hours, program compute it during several minutes. The work on the algorithm is not finished, as for $\psi_k(\varphi)$ with larger $k$ we have a choice of many expressions to integrate by parts, and **Test** procedure must be done much more complicated.

# Chapter 4

# Center and Composition Conditions for Abel differential equations with $p$, $q$ – trigonometric polynomials of small degrees

## 4.1 Introduction

As it was shown in [Ch1], starting from homogeneous perturbation $F_d(x, y)$, $G_d(x, y)$ of degree $d$, we obtain Abel equation with $p$, $q$ – homogeneous polynomials in $\sin x$, $\cos x$ of degrees $d+1$, $2d+2$ respectively. It means that the lowest degrees of trigonometric polynomials in (1.2), related to dynamical system (1.1), are 3 and 6 respectively. The center conditions for them are known, and they are not restricted to the composition conditions.

One can consider Abel equation

$$y' = p(x)y^2 + q(x)y^3 \tag{4.1}$$

with $p(x)$, $q(x)$ – arbitrary trigonometric polynomials, i.e. polynomials of any degree. As before, **Center Condition for Abel equation** is periodicity of solutions $y(x)$ between two specified points 0 and $2\pi$, i.e. $y(0) = y(2\pi)$ for all solutions $y(x)$ starting at 0 with sufficiently small initial value $y(0)$.

As before, **composition condition** in the form $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ for some function $W(x)$ with $W(0) = W(2\pi)$ is **sufficient condition** for the center.

The question "when center and composition conditions coincide" can be stated for any functions $p$, $q$. For $p(x)$, $q(x)$ – algebraic polynomials in $x$ of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ this question was studied in [BFY2-3, BY1], and the following **Composition Conjecture** was stated: *for algebraic polynomials $p$, $q$ composition condition is not only sufficient, but also necessary for the center.* The computational verification for $p$, $q$ – algebraic polynomials of small degrees was done in [BY1].

We consider trigonometric polynomials in a general form

$$p(\varphi) = \lambda_0 + \sum_{k=1}^{\deg p(\varphi)} (\lambda_{2k-1}\sin(k\varphi) + \lambda_{2k}\cos(k\varphi)). \tag{4.2}$$

Such representation is unique, since the functions $1, \cos\varphi, \sin\varphi, \cos 2\varphi, \sin 2\varphi, \ldots$ form a basis in the space of all trigonometric polynomials. As the **degree of trigonometric polynomial** the maximal $d$ was considered, such that $\lambda_{2d}$ or $\lambda_{2d-1}$ is not equal to zero. As **free term** we consider $\lambda_0$.

In this chapter we show that for small degrees of trigonometric polynomials $p$, $q$ (up to 2) center condition implies composition condition. All these composition conditions will be directly written down. Part of computations was done together with M. Kiermaier, working on this summer project under my supervision.

## 4.2 Computational approach

We proved the following theorem:

**Theorem.** *For $p$, $q$ up to degrees 2, composition condition is not only sufficient, but necessary for the center.*

To prove it we will compute subsequently coefficients $\psi_k(x)$ of Poincare return map

$$y(x, y_0) = y_0 + \sum_{k=2}^{\infty} \psi_k(x, \lambda) y_0^k,$$

and after substitution $x = 2\pi$ we will look for conditions on coefficients $\lambda_i$, $\mu_i$ under which $\psi_k(2\pi) = 0$. The integration of trigonometric functions works very slow in "Mathematica", so to increase the speed of computations a function `integ` was introduced, which replaced the standard `Integrate` procedure of Mathematica:

```
$RecursionLimit=2000;
integ[y_ + z_,x_]:=integ[y,x]+integ[z,x];
integ[c_ y_,x_]:=c integ[y,x] /;FreeQ[c,x];
integ[c_,x_]:=c x /;FreeQ[c,x];
integ[x_^n_.,x_]:=x^(n+1)/(n+1) /;FreeQ[n,x]&& n != -1;
integ[Sin[a_ x_],x_]:=-Cos[a x]/a +1/a  /;FreeQ[a,x];
integ[Cos[a_ x_],x_]:= Sin[a x]/a /;FreeQ[a,x];
integ[Sin[x_],x_]:=-Cos[x] +1 ;
integ[Cos[x_],x_]:= Sin[x] ;
```

Then we use `TrigReduce` to reduce trigonometric expressions to the form where `integ` can be applied.

```
psi[x_][i]=
TrigReduce[integ[TrigReduce[-(i-1)*psi[x][i-1]*p[x]-(i-2)*psi[x][i-2]*q[x]],x]];
```

After computing first several equations $\psi_k(2\pi) = 0$ we prove that these conditions imply composition representability of $P(\varphi)$, $Q(\varphi)$, and hence imply center.

Let us fix notations we work with. Let us notice, that integrating (4.2) w.r.t. $\varphi$ and taking into consideration $\int_0^{2\pi} \sin t \, dt = 0$, $\int_0^{2\pi} \cos t \, dt = 0$ we got that $p(\varphi)$, $q(\varphi)$ do not have free terms, i.e

$$\begin{cases} p(\varphi) = \displaystyle\sum_{i=1}^{\deg p(\varphi)} (\lambda_{2i-1} \cos(\varphi) - \lambda_{2i} \sin(\varphi)) \\ q(\varphi) = \displaystyle\sum_{i=1}^{\deg q(\varphi)} (\mu_{2i-1} \cos(\varphi) - \mu_{2i} \sin(\varphi)), \end{cases}$$

which corresponds to the following $P$, $Q$:

$$\begin{cases} P(\varphi) = \sum_{i=1}^{\deg p(\varphi)} (\lambda_{2i-1}\sin(\varphi) + \lambda_{2i}\cos(\varphi) - \lambda_{2i}) \\ Q(\varphi) = \sum_{i=1}^{\deg q(\varphi)} (\mu_{2i-1}\sin(\varphi) + \mu_{2i}\cos(\varphi) - \mu_{2i}) \end{cases}$$

Notice, that in contrast to the polynomial case, the degree of trigonometric polynomial does not change after integration or differentiation.

## 4.3 Center conditions for $p$, $q$ of small degrees

### 4.3.1 Center conditions for the case when either $p$ or $q$ is of the degree zero

If degree $p$ is equal to zero, then $p$ is a constant, and as free term of $p$ is zero, $p \equiv 0$. Then the equation (4.1) becomes

$$y' = q_a(x)y^3.$$

**Lemma 4.3.1** *The condition $Q(2\pi) = 0$ is necessary and sufficient for the existence of a center.*

**Proof:** This equation can be solved explicitly:

$$\frac{y'}{y^3} = q(x), \quad \left(\frac{1}{-2y^2}\right)' = (Q(x))', \quad \frac{1}{2y^2} + Q(x) = Const,$$

substituting $x = 0$ we get $Const = \dfrac{1}{2y(0)^2}$ Now it's obvious, that the center condition $y(0) = y(2\pi)$ is equivalent to $Q(2\pi) = 0$. ■

If the degree of $q$ is equal to one, then similarly we obtain the center condition is $P(2\pi) = 0$.

**Corollary:** *If $p \equiv 0$ ($q \equiv 0$), any trigonometric polynomial $q$ ($p$, respectively) without free term defines center. Formally speaking, there exists a composition with polynomial $q$ ($p$, respectively).*

### 4.3.2 Center conditions for the case $\deg p(\varphi) = 1$, $\deg q(\varphi) = 1$.

**Lemma 4.3.2** *For* $\deg p(\varphi) = \deg q(\varphi) = 1$ *center conditions coincide with the composition conditions, i.e. there exists center iff* $P(\varphi) = kQ(\varphi)$.

**Proof:** We get:

$$\begin{cases} P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2 \\ Q(\varphi) = \mu_1 \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 \end{cases}$$

After computing we get:
$\psi_4(2\pi) = \pi(-\lambda_2\mu_1 + \lambda_1\mu_2) = 0$ only if $\lambda_1\mu_2 = \lambda_2\mu_1$, therefore this condition is necessary for center. Hence $\lambda_1 = k\mu_1$, $\lambda_2 = k\mu_2$ for some $k \neq 0$, hence $P(\varphi) = kQ(\varphi)$. ■

### 4.3.3 Center conditions for the case $\deg p(\varphi) = 1$, $\deg q(\varphi) = 2$.

**Lemma 4.3.3** *For* $\deg p(\varphi) = 1$, $\deg q(\varphi) = 2$ *center conditions coincide with the composition conditions. There are three center components for this case:*

$$\lambda_1 = 0,\ \mu_1 = 0,\ \mu_3 = 0 \qquad (1.2.A),$$

*corresponding to the composition with* $W(\varphi) = \cos\varphi - 1$,

$$\lambda_2 = 0,\ \mu_2 = 0,\ \mu_3 = 0 \qquad (1.2.B),$$

*corresponding to the composition with* $W(\varphi) = \sin\varphi$,

$$\mu_1 = \frac{\lambda_1\mu_2}{\lambda_2},\ \mu_4 = \frac{\mu_3(\lambda_2^2 - \lambda_1^2)}{2\lambda_1\lambda_2} \qquad (1.2.C),$$

*corresponding to the composition*

$$Q(\varphi) = \frac{\mu_3}{\lambda_1\lambda_2}P(\varphi)^2 + \frac{\lambda_1\mu_2 + 2\lambda_2\mu_3}{\lambda_1\lambda_2}P(\varphi).$$

**Proof:** We get:

$$\begin{cases} P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2 \\ Q(\varphi) = \mu_1 \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 + \mu_3 \sin(2\varphi) + \mu_4 \cos(2\varphi) - \mu_4 \end{cases}$$

After computing we get:
$\psi_4(2\pi) = \pi(-\lambda_2\mu_1 + \lambda_1\mu_2)$
$\psi_5(2\pi) = \pi(2\lambda_2^2\mu_1 - 2\lambda_1\lambda_2\mu_2 + \lambda_1^2\mu_3 - \lambda_2^2\mu_3 + 2\lambda_1\lambda_2\mu_4)$
$\psi_6(2\pi) = \frac{1}{4}\pi(-3\lambda_1^2\lambda_2\mu_1 - 15\lambda_2^3\mu_1 + 3\lambda_1^3\mu_2 + 15\lambda_1\lambda_2^2\mu_2 + 4\lambda_2\mu_1\mu_2 - 4\lambda_1\mu_2^2 - 12\lambda_1^2\lambda_2\mu_3 + 12\lambda_2^3\mu_3 + 2\lambda_1\mu_1\mu_3 - 2\lambda_2\mu_2\mu_3 - 24\lambda_1\lambda_2^2\mu_4 + 6\lambda_2\mu_1\mu_4 - 2\lambda_1\mu_2\mu_4)$.

We assume that $(\lambda_1 \neq 0 \vee \lambda_2 \neq 0) \wedge (\mu_3 \neq 0 \vee \mu_4 \neq 0)$.

$\psi_4 = 0$ if either
**Case 1:** $\mu_1 = \dfrac{\lambda_1\mu_2}{\lambda_2}$ and $\lambda_2 \neq 0$.
or
**Case 2:** $\lambda_2 = 0$ . In this case $\lambda_1 = \lambda_2 = 0$ is not a solution because we get $\deg p = 0$, so we left with the only possibility $\mu_2 = 0$.

Consider the first case. Computing $\psi_5(2\pi) = 0$ under the written relation, we get $\psi_3(2\pi) = \psi_4(2\pi) = 0$ and $\psi_5(2\pi) = \pi(2\lambda_1\lambda_2\mu_4 - \mu_3(\lambda_2^2 - \lambda_1^2))$.
Solving the equation $\psi_5(2\pi) = 0$, we get two cases:

**Case 1.1** $\mu_1 = \dfrac{\lambda_1\mu_2}{\lambda_2}$, $\mu_4 = \dfrac{\mu_3(\lambda_2^2 - \lambda_1^2)}{2\lambda_1\lambda_2}$ and $\lambda_1 \neq 0 \wedge \lambda_2 \neq 0$.

**Case 1.2** $\lambda_1 = \mu_1 = \mu_3 = 0$.

**Case 2.** Computing $\psi_5(2\pi) = 0$, we get $\lambda_2 = \mu_2 = \mu_3 = 0$.

$\psi_6 = 0$ for all the three cases. Hence we have obtained three sets of conditions on coefficients, which are suspicious of being center conditions. Let's analyze them.

**Case 1.2**
In this case we get $P(\varphi) = \lambda_2 \cos\varphi - \lambda_2$,
$Q(\varphi) = \mu_2 \cos\varphi - \mu_2 + \mu_4 \cos 2\varphi - \mu_4 = \mu_2 \cos\varphi - \mu_2 + \mu_4(2\cos^2\varphi - 2) = \mu_2(\cos\varphi - 1) + 2\mu_4((\cos\varphi - 1)^2 + 2(\cos\varphi - 1)) = (\mu_2 + 4\mu_4)(\cos\varphi - 1) +$

$2\mu_4(\cos\varphi - 1)^2$.
So there is a composition with $W(\varphi) = \cos\varphi - 1$, $\tilde{P}(W) = \lambda_2 W$ and $\tilde{Q}(W) = (\mu_2 + 4\mu_4)W + 2\mu_4 W^2$, so this condition is a composition condition and therefore this condition is sufficient for center.

**Case 2**
In this case we get $P(\varphi) = \lambda_1 \sin\varphi$,
$Q(\varphi) = \mu_1 \sin\varphi + \mu_4 \cos 2\varphi - \mu_4 = \mu_1 \sin\varphi + \mu_4(-2\sin^2\varphi + 1) - \mu_4 = \mu_1 \sin\varphi - 2\mu_4 \sin^2\varphi$.
So, in this case we get $P$ and $Q$ are compositions with $W(\varphi) = \sin\varphi$, this condition is composition condition and hence it is sufficient for the existence of a center.

**Case 1.1**
In this case we get
$P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2$
$Q(\varphi) = \dfrac{\lambda_1\mu_2}{\lambda_2}\sin(\varphi) + \mu_2\cos(\varphi) - \mu_2 + \mu_3\sin(2\varphi) + \dfrac{\mu_3(\lambda_2^2 - \lambda_1^2)}{2\lambda_1\lambda_2}\cos(2\varphi) - \dfrac{\mu_3(\lambda_2^2 - \lambda_1^2)}{2\lambda_1\lambda_2}$
Let's look for the composition condition $Q(\varphi) = a\,P^2(\varphi) + b\,P(\varphi)$;

Substituting $P$ and $Q$ into this equation, we get:
$(a\lambda_1\lambda_2 - \mu_3)\sin(2x) + (-\frac{1}{2}a\lambda_1^2 + \frac{1}{2}a\lambda_2^2 - \mu_4)\cos(2x) + (b\lambda_1 - 2a\lambda_1\lambda_2 - \mu_1)\sin(x) + (b\lambda_2 - 2a\lambda_2^2 - \mu_2)\cos(x) + (\frac{1}{2}a\lambda_1^2 - b\lambda_2 + \frac{3}{2}a\lambda_2^2 - \mu_2 - \mu_4) = 0$

Solving this system, we obtain:

$$a = \frac{\mu_3}{\lambda_1\lambda_2}, \quad b = \frac{\lambda_1\mu_2 + 2\lambda_2\mu_3}{\lambda_1\lambda_2}$$

For this values of $a$ and $b$ we verified that $a\,P^2(\varphi) + b\,P(\varphi) - Q(\varphi) \equiv 0$, and therefore in the case 1.1 center condition coincide with the composition condition. ■

### 4.3.4 Center conditions for the case $\deg p(\varphi) = 2$, $\deg q(\varphi) = 1$

**Lemma 4.3.4** *For $\deg p(\varphi) = 2$, $\deg q(\varphi) = 1$ center conditions coincide with the composition conditions. There are four center components for this case:*

$$\lambda_1 = 0,\ \lambda_3 = 0,\ \mu_1 = 0, \tag{2.1.A}$$

*corresponding to the composition with $W(\varphi) = \cos\varphi - 1$,*

$$\lambda_2 = 0,\ \lambda_3 = 0,\ \mu_2 = 0, \tag{2.1.B}$$

*corresponding to the composition with $W(\varphi) = \sin\varphi$, and*

$$\lambda_1 = \frac{\lambda_2\mu_1}{\mu_2},\ \lambda_4 = \frac{\lambda_3(\mu_2^2 - \mu_1^2)}{2\mu_1\mu_2}, \tag{2.1.C}$$

*corresponding to the composition*

$$P(\varphi) = \frac{\lambda_3}{\mu_1\mu_2}Q^2(\varphi) + \frac{\mu_1\lambda_2 + 2\mu_2\lambda_3}{\mu_1\mu_2}Q(\varphi),$$

*Exchanging $\lambda_k$ and $\mu_k$, we get that these representation coincides with the composition representation for $\deg p(\varphi) = 1$, $\deg q(\varphi) = 2$. As the composition condition is symmetric for these two cases, so the composition representations must be symmetric as well.*

**Proof:** We get:

$$\begin{cases} P(\varphi) = \lambda_1\sin(\varphi) + \lambda_2\cos(\varphi) - \lambda_2 + \lambda_3\sin(2\varphi) + \lambda_4\cos(2\varphi) - \lambda_4 \\ Q(\varphi) = \mu_1\sin(\varphi) + \mu_2\cos(\varphi) - \mu_2 \end{cases}$$

After computing we get:

$\psi_4(\pi) = \pi(-\lambda_2\mu_1 + \lambda_1\mu_2)$
$\psi_5(\pi) = \pi(2\lambda_2^2\mu_1 - \lambda_1\lambda_3\mu_1 + \lambda_2\lambda_4\mu_1 - 2\lambda_1\lambda_2\mu_2 + \lambda_2\lambda_3\mu_2 - 3\lambda_1\lambda_4\mu_2)$
$\psi_6(\pi) = \frac{1}{4}\pi(-3\lambda_1^2\lambda_2\mu_1 - 15\lambda_2^3\mu_1 + 12\lambda_1\lambda_2\lambda_3\mu_1 - 6\lambda_2\lambda_3^2\mu_1 - 12\lambda_2^2\lambda_4\mu_1 + 12\lambda_1\lambda_3\lambda_4\mu_1 -$
$6\lambda_2\lambda_4^2\mu_1 - 2\lambda_3\mu_1^2 + 3\lambda_1^3\mu_2 + 15\lambda_1\lambda_2^2\mu_2 - 12\lambda_2^2\lambda_3\mu_2 + 6\lambda_1\lambda_3^2\mu_2 + 36\lambda_1\lambda_2\lambda_4\mu_2 -$

$12\lambda_2\lambda_3\lambda_4\mu_2 + 30\lambda_1\lambda_4^2\mu_2 + 4\lambda_2\mu_1\mu_2 - 4\lambda_4\mu_1\mu_2 - 4\lambda_1\mu_2^2 + 2\lambda_3\mu_2^2)$

We assume that $(\lambda_3 \neq 0 \vee \lambda_4 \neq 0)$ and $(\mu_1 \neq 0 \vee \mu_2 \neq 0)$.

$\psi_4(\pi) = 0$ iff at least one of the three conditions is satisfied:

**Case 1.** $\lambda_1 = \dfrac{\lambda_2\mu_1}{\mu_2} \wedge \mu_2 \neq 0$

**Case 2.** $\lambda_2 = \mu_2 = 0$

Now we compute $\psi_5(2\pi) = 0$ for these cases:

In the case 1 we get $\psi_5(2\pi) = \dfrac{\lambda_2\pi}{\mu_2}(-2\lambda_4\mu_1\mu_2 + \lambda_3\mu_2^2 - \lambda_3\mu_1^2)$, so we obtain the following cases:

**Case 1.1** $\lambda_4 = \dfrac{\lambda_3(\mu_2^2 - \mu_1^2)}{2\mu_1\mu_2}$, $\lambda_1 = \dfrac{\lambda_2\mu_1}{\mu_2}$, $\mu_1 \neq 0$, $\mu_2 \neq 0$.

Here
$P(\varphi) = \dfrac{\lambda_2\mu_1}{\mu_2}\sin(\varphi) + \lambda_2\cos(\varphi) - \lambda_2 + \lambda_3\sin(2\varphi) + \dfrac{\lambda_2\mu_1}{\mu_2}\cos(2\varphi) - \dfrac{\lambda_2\mu_1}{\mu_2}$,
$Q(\varphi) = \mu_1\sin(\varphi) + \mu_2\cos(\varphi) - \mu_2$,
solving the equation $P(x) - a\,Q(x)^2 - b\,Q(x) = 0$ w.r.to $a, b$, we found $a = \dfrac{\lambda_3}{\mu_1\mu_2}$, $b = \dfrac{\lambda_2\mu_1 + 2\lambda_3\mu_2}{\lambda_1\lambda_2}$, and for these values of $a, b$ $P(x) - a\,Q(x)^2 - b\,Q(x) \equiv 0$. So, this case is composition case.

**Case 1.2** $\lambda_1 = \lambda_3 = \mu_1 = 0$ Then
$P(\varphi) = \lambda_2\cos\varphi - \lambda_2 + \lambda_4\cos 2\varphi - \lambda_4 = 2\lambda_4(cos(\varphi) - 1)^2 + (\lambda_2 + 44l_4)(cos(\varphi) - 1)$, $Q(\varphi) = \mu_2(\cos\varphi - 1)$, so we get composition with $W(\varphi) = \cos(\varphi) - 1$, hence it is center condition.

In the case 2 we get
$\psi_5(2\pi) = -\pi\lambda_1\lambda_3\mu_1$, $\psi_6(2\pi) = \dfrac{1}{2}\pi\lambda_3\mu_1(20\lambda_1^2 + 40\lambda_1\lambda_3 - 4\lambda_1\lambda_4 - m_1)$, and hence from $\psi_5(2\pi) = 0$ follows $\lambda_1 = 0 \vee \lambda_3 = 0 \vee \mu_1 = 0$. From the third solution follows $p(\varphi) = 0$, hence this case is impossible, and we are left with the two cases:

**Case 2.1.** $\lambda_2 = \lambda_3 = \mu_2 = 0$, and in this case also $\psi_6(2\pi) = 0$ and we get
$P(\varphi) = \lambda_1 \sin\varphi + \lambda_4 \cos 2\varphi - \lambda_4 = -2\lambda_4 \sin^2\varphi + \lambda_1 \sin\varphi$, $Q(\varphi) = \mu_1 \sin\varphi$, so $P$ and $Q$ are compositions with $W(\varphi) = \sin\varphi$.

**Case 2.2.** $\lambda_1 = \lambda_2 = \mu_2 = 0$, and then from $\psi_6(2\pi) = 0$ follows $\lambda_3 = 0$, so this case coincide with 2.1. ■

### 4.3.5 Center conditions for the case $\deg p(\varphi) = 2$, $\deg q(\varphi) = 2$.

In this case $P$ and $Q$ are of the form

$$P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2 + \lambda_3 \sin(2\varphi) + \lambda_4 \cos(2\varphi) - \lambda_4$$

$$Q(\varphi) = \mu_1 \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 + \mu_3 \sin(2\varphi) + \mu_4 \cos(2\varphi) - \mu_4$$

Without loss of generality we can shift coefficients $p$ and $q$ of the Abel equation on the angle $\alpha$:
$p(x) -> p(x+\alpha)$,
$q(x) -> q(x+\alpha)$.
Then
$P_{new}(\varphi) = \int_0^{\varphi} p_{new}(x)dx = \int_0^{\varphi} p(x+\alpha)dx = P(x+\alpha) - P(\alpha)$,
and similarly $Q$.

Shift on the angle $\alpha$:
$P(x+\alpha) - P(\alpha) = -2\cos\alpha\sin\alpha + 2\cos\alpha\sin\alpha\cos 2x - \sin\alpha\lambda_1 + \sin\alpha\cos x\lambda_1 - \cos\alpha\lambda_2 + \cos\alpha\cos x\lambda_2 - \cos^2\alpha\lambda_4 + \sin^2\alpha\lambda_4 + \cos^2\alpha\cos 2x\lambda_4 - \sin^2\alpha\cos 2x\lambda_4 + \cos\alpha\lambda_1\sin x - \sin\alpha\lambda_2\sin x + \cos^2\alpha\sin 2x - \sin^2\alpha\sin 2x - 2\cos\alpha\sin\alpha\lambda_4\sin 2x = \cos x(\lambda_1\sin\alpha + \lambda_2\cos\alpha) + \cos 2x(2\cos\alpha\sin\alpha + \lambda_4\cos^2\alpha - \lambda_4\sin^2\alpha) + \ldots$,
so we can kill any term by a shift on an appropriate angle $\alpha$.

Using rotation, we can put any coefficient being equal to zero, let $\lambda_3 = 0$, then using rescaling, we get $\lambda_4 = 1$ (both of them can not be zeroes, since the degree of $P$ is 2).

**Lemma 4.3.5** *For*

$$\begin{cases} P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2 + \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_1 \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 + \mu_3 \sin(2\varphi) + \mu_4 \cos(2\varphi) - \mu_4 \end{cases}$$

*the necessary and sufficient conditions for the center is either $P$ is proportional to $Q$, or both of them are functions only of $\sin\varphi$ or $\cos\varphi$.*

**Proof:** From $\psi_4(2\pi) = \pi(-\lambda_2\mu_1 + \lambda_1\mu_2 - 2\mu_3) = 0$, we find $\mu_3 = \frac{1}{2}(\lambda_1\mu_2 - \lambda_2\mu_1)$, substituting it into $\psi_5(2\pi)$ we get
$\psi_5(2\pi) = 1/2\pi(\lambda_2^3\mu_1 + \lambda_1(-2+\lambda_1^2))\mu_2 - \lambda_1\lambda_2^2\mu_2 - \lambda_2(((2+\lambda_1^2)\mu_1 - 4\lambda_1\mu_4))$

Solving it, we have several options.

**Case I.** $\lambda_1 = 0$, then $\psi_5(2\pi) = \frac{1}{2}\pi\lambda_2\mu_1(\lambda_2^2 - 2)$, so either $\lambda_2 = 0$ (**Case I.1**), or $\mu_1 = 0$ (**Case I.2**), or $\lambda_2 = \pm\sqrt{2}$ (**Case I.3**).

**Case II.** $\lambda_2 = 0$, then $\psi_5(2\pi) = \frac{1}{2}\pi\lambda_1\mu_2(\lambda_1^2 - 2))$, so either $\lambda_1 = 0$ (**Case II.1**), or $\mu_2 = 0$ (**Case II.2**), or $\lambda_1 = \pm\sqrt{2}$ (**Case II.3**). Case II.1. coincides with the Case I.1.

**Case III.** $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, then from $\psi_5(2\pi) = 0$ we get

$$\mu_4 = -\frac{1}{4\lambda_1\lambda_2}(-(2+\lambda_1^2)\lambda_2\mu_1 + \lambda_2^3\mu_1 + \lambda_1(-2+\lambda_1^2)\mu_2 - \lambda_1\lambda_2^2\mu_2)$$

Substituting it into the program, we get

$$\psi_6(2\pi) = -\frac{\pi}{8\lambda_1\lambda_2}(\lambda_2^2(-2+\lambda_2^2)\mu_1^2 - 2\lambda_1^3\lambda_2\mu_1(3\lambda_2 + \mu_2) + \lambda_1^4\mu_2(6\lambda_2 + \mu_2) - 2\lambda_1\lambda_2\mu_1(-3\lambda_2 + 3\lambda_2^3 - 2\mu_2 + \lambda_2^2\mu_2) + \lambda_1^2(-6\lambda_2\mu_2 + 6\lambda_2^3\mu_2 - 2\mu_2^2 + \lambda_2^2(\mu_1^2 + \mu_2^2))) =$$

$$-\frac{\pi}{8}(\mu_1 - \frac{\lambda_1\mu_2}{\lambda_2})(\lambda_1^2 + \lambda_2^2 - 2)\left(\mu_1 - \frac{-\lambda_1^3(6\lambda_2 + \mu_2) + \lambda_1(6\lambda_2 - 6\lambda_2^3 + 2\mu_2 - \lambda_2^2\mu_2)}{\lambda_2(-2 + \lambda_1^2 + \lambda_2^2)}\right).$$

Solving the equation $\psi_6(2\pi) = 0$ with respect to $\mu_1$, we get the following cases:

**Case III.1** $\mu_1 = \dfrac{\lambda_1\mu_2}{\lambda_2}$, and this is the only solution if $\lambda_1^2 + \lambda_2^2 = 2$.
**Case III.2** If $\lambda_1^2 + \lambda_2^2 \neq 2$, there is another solution

$$\mu_1 = \frac{-\lambda_1^3(6\lambda_2 + \mu_2) + \lambda_1(6\lambda_2 - 6\lambda_2^3 + 2\mu_2 - \lambda_2^2\mu_2)}{\lambda_2(-2 + \lambda_1^2 + \lambda_2^2)}.$$

In the Case III.2 we get
$\psi_7(2\pi) = \dfrac{3\pi\lambda_1(-1 + \lambda_1^2 + \lambda_2^2)}{(-2 + \lambda_1^2 + \lambda_2^2)^2}((6 - 13\lambda_1^2 + 8\lambda_1^4)\lambda_2 + (-11 + 12\lambda_1^2)\lambda_2^3 + 4\lambda_2^5 +$
$2(2 - 3\lambda_1^2 + \lambda_1^4)\mu_2 + 2(-3 + 2\lambda_1^2)\lambda_2^2\mu_2 + 2\lambda_2^4\mu_2)$.

We can find $\mu_2$ from the equality $\psi_7(2\pi) = 0$, it is possible only for

$$\mu_2 = -\frac{(6 - 13\lambda_1^2 + 8\lambda_1^4)\lambda_2 + (-11 + 12\lambda_1^2)\lambda_2^3 + 4\lambda_2^5}{2(2 - 3\lambda_1^2 + \lambda_1^4) + 2(-3 + 2\lambda_1^2)\lambda_2^2 + 2\lambda_2^4}.$$

For these values of $\mu$ and $\lambda$ all polynomials up to $\psi_7(2\pi)$ vanish, and we compute $\psi_8(2\pi)$, $\psi_9(2\pi)$,$\psi_{10}(2\pi)$.
After standard computation of resultants (see [B1] or Chapter 5 for short explanation of resultants technique), we obtain that these three polynomials do not have common zeroes.

Let's consider the rest of all of these cases.
In the Case I.1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1$, then $\mu_3 = 0$, and

$$\begin{cases} P(\varphi) = \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_1\sin(\varphi) + \mu_2\cos(\varphi) - \mu_2 + \mu_4\cos(2\varphi) - \mu_4 \end{cases},$$

computing $\psi_6(2\pi)$ we get either $\mu_1 = 0$, or $\mu_2 = 0$, and in both cases we get the composition - with $\cos\varphi$ and $\sin\varphi$ respectively.
In the Case I.2. $\lambda_1 = \mu_1 = \lambda_3 = 0$, $\lambda_4 = 1$, then $\mu_3 = 0$,

$$\begin{cases} P(\varphi) = \lambda_2\cos(\varphi) - \lambda_2 + \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_2\cos(\varphi) - \mu_2 + \mu_4\cos(2\varphi) - \mu_4 \end{cases},$$

and we get composition with $\cos\varphi$.
In the Case I.3. $\lambda_1 = \lambda_3 = 0$, $\lambda_4 = 1$, $\lambda_2 = \sqrt{2}$, (case $\lambda_2 = -\sqrt{2}$ is similar)

$$\begin{cases} P(\varphi) = \sqrt{2}\cos(2\varphi) - \sqrt{2} + \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_1\sin(\varphi) + \mu_2\cos(\varphi) - \mu_2 + \mu_3\sin(2\varphi) + \mu_4\cos(2\varphi) - \mu_4 \end{cases}.$$

Performing computations, we get $\mu_1 = 0$, and it is a composition with $\cos\varphi$.

In the Case II.2. $\lambda_2 = \mu_2 = \lambda_3 = 0$, $\lambda_4 = 1$, then $\mu_3 = 0$,

$$\begin{cases} P(\varphi) = \lambda_1 \sin(2\varphi) + \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_1 \sin(\varphi) + \mu_4 \cos(2\varphi) - \mu_4 \end{cases},$$

and we get composition with $\sin\varphi$.

In the Case II.3. $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1$, $\lambda_1 = \sqrt{2}$, (case $\lambda_2 = -\sqrt{2}$ is similar)

$$\begin{cases} P(\varphi) = \sqrt{2}\sin(2\varphi) + \cos(2\varphi) - 1 \\ Q(\varphi) = \mu_1 \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 + \mu_3 \sin(2\varphi) + \mu_4 \cos(2\varphi) - \mu_4 \end{cases}.$$

Performing computations, we get $\mu_2 = 0$, and it is a composition with $\sin\varphi$.

In the Case III.1. $\lambda_3 = 0$, $\lambda_4 = 1$, $\mu_1 = \dfrac{\lambda_1 \mu_2}{\lambda_2}$, and then $\mu_3 = 0$, so

$$\begin{cases} P(\varphi) = \lambda_1 \sin(\varphi) + \lambda_2 \cos(\varphi) - \lambda_2 + \cos(2\varphi) - 1 \\ Q(\varphi) = \frac{\lambda_1 \mu_2}{\lambda_2} \sin(\varphi) + \mu_2 \cos(\varphi) - \mu_2 + \mu_4 \cos(2\varphi) - \mu_4 \end{cases}.$$

Then from $\psi_5(2\pi) = 0$ we get either $\lambda_1 = 0$, and then $P$, $Q$ are compositions with $\cos\varphi$, or $\mu_2 = \lambda_2 \mu_4$, then $\mu_1 = \lambda_1 \mu_4$, and $Q = \mu_4 P$, i.e. we again get composition. ■

# Chapter 5

# Center and Composition Conditions for Abel differential equations with $p$, $q$ – algebraic polynomials of small degrees

## 5.1 Introduction

In this chapter we summarize results on Abel equation with $p$, $q$ – algebraic polynomials, which were obtained in [BFY1-3, BY1], in the authors M.Sc. thesis and in the current Ph.D. thesis. All the results, for which an explicit reference is not given, are new and belong to the current thesis, so they are given with proofs.

We consider Abel differential equation

$$y' = p(x)y^2 + q(x)y^3 \tag{5.1}$$

with $p(x)$, $q(x)$ – algebraic polynomials in $x$. Although this problem does not correspond directly to the classical center problem on a plane (where $p$, $q$ are trigonometric polynomials), it represents a significant interest by itself, as it helps to understand the combinatorial structure of the set of closed solutions. A center condition for this equation (closely related to the classical center condition for polynomial vector fields on the plane) is that for fixed points 0 and $a$ and for any solution $y(x)$ $y_0 = y(0) \equiv y(a)$ , i.e. all the solutions return to the initial value on the interval $[0, a]$.

As before, we are looking for solutions of (5.1) in the form

$$y(x, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(x, \lambda) y_0^k,$$

where $y(0, y_0) = y_0$, $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the (finite) set of the coefficients of $p$, $q$. Shortly we will write $v_k(x)$.

The coefficients $v_k$ are computed by formula (3.4) and turn out to be polynomials both in $x$ and $\lambda$. We study the polynomials ideals $I_k \subseteq \mathbb{R}[\lambda\ , x]$, $I_k = \{v_2(x), v_3(x), \ldots v_k(x)\}$. The problem is **to find conditions on $p$, $q$, under which $x = a$ is a common zero of all $I_k$.**

Following [BFY2] we introduce periods of the equation (5.1) as those $\omega \in \mathbb{C}$, for which $y(0) \equiv y(\omega)$ for any solution $y(x)$ of (5.1). The **generalized center conditions** are conditions on $p$, $q$ under which given $a_1, \ldots, a_k$ are (exactly all) the periods of (5.1).

The ideal $I \subseteq \mathbb{C}[\lambda,\ x]$ is studied, where $\lambda$ is a set of coefficients of polynomials $p$, $q$.

$$I = \{v_2(x), v_3(x), \ldots v_k(x), \ldots\} = \bigcup_{k=2}^{\infty} I_k, \text{ where } I_k = \{v_2(x), v_3(x), \ldots v_k(x)\}.$$

Our **generalized center problem** is the following:
*For a given set of different complex numbers $a_1 = 0, a_2, \ldots, a_\ell$ find conditions on algebraic polynomials $p$, $q$, under which these numbers are common zeroes of $I$.*
We say that (5.1) defines **center on the set of numbers** $a_2, \ldots, a_\ell$ . These numbers are called **periods** of (5.1), since $y(0) = y(a_i)$ for all the solutions $y(x)$ of (5.1).

## 5.2 Composition conjecture for multiple zeroes and its status for small degrees of $p$ and $q$

The following **composition conjecture** has been proposed in [BFY2]:
$I = \bigcup_{k=1}^{\infty} I_k$ **has zeroes** $a_1, a_2, \ldots a_k$ , $a_1 = 0$, **if and only if**

$$P(x) = \int_0^x p(t)dt = \tilde{P}(W(x)) \ , \ Q(x) = \int_0^x q(t)dt = \tilde{Q}(W(x)) \ ,$$

**where** $W(x) = \prod_{i=1}^{k} (x - a_i)\tilde{W}(x)$ **is a polynomial, vanishing at** $a_1, a_2, \ldots a_k$**, and** $\tilde{P}$**,** $\tilde{Q}$ **are some polynomials without free terms (**$\tilde{W}(x)$**is an arbitrary polynomial).**

Sufficiency of this conjecture can be shown easily (see [BFY2]). But we still do not have any method to prove the necessity of this conjecture in the general case, although the connection between this conjecture and some interesting analytic problems was established (see [BFY1], [BFY2], [BFY3]), and for some simplified cases it was partially or completely proved.

The following work was started during M.Sc. thesis [B1] and was completed during Ph.D. thesis:

a) The maximal number of different zeroes of $I$, i.e. the maximal number of periods of (5.1) was estimated:

**Theorem 5.2.1** *Either the number of surviving different zeroes (including* $0$*) of* $I$ *is less or equal then* $(\deg P + \deg Q)/3$*, or* $P$ *is proportional to* $Q$*.*

**Corollary:** *Either* $P$ *is proportional to* $Q$*, or the number of different periods of (1.2) is less than or equal to* $((\deg P + \deg Q)/3) - 1$*.*

This result was proved in M.Sc. thesis and is implicitly contained in computations, given in [BFY3] .

The next results were obtained in M.Sc. thesis [B1, BY1].

b) The above stated composition conjecture was verified for the following cases:
$(\deg P, \deg Q) = (2,7), (3,4), (4,2), (4,3), (4,4), (5,2), (6,2), (3,6)$. It was done using computer symbolic calculations with some convenient representation of $P$ and $Q$ . Computations for the cases $(\deg P, \deg Q) = (2,7), (3,6)$ were performed together with Jonatan Gutman and Carla Scapinello, working on this summer project under my supervision.

Namely, the following theorem was proved:

**Theorem.** *For the following cases the composition conjecture is true and the following table gives the possible number of different periods in each case:*

| deg P / deg Q | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | 0, 1 | 0 | 0, 1 | 0 | 0, 1 |
| 3 | 0 | 0, 2 | 0 | | |
| 4 | 0, 1 | 0 | 0, 1, 3 | | |
| 5 | 0 | | | | |
| 6 | 0, 1 | 0, 2 | | | |
| 7 | 0 | | | | |

The proof consists of computations of several first $\psi_n(x)$ for each of the cases considered, equating $\psi_n(x)$ to zero and solving the resulting systems of polynomial equations on coefficients of $p$ and $q$ and on $a$. It was done using computer symbolic calculations using the special representation of $P$ and $Q$. In most of the cases straightforward computations were far beyond the limitations of the computer used. Consequently, some non-obvious analytic simplifications were used.

For computations we used resultants technique. Resultants provide us with a convenient tool for checking, whether $n+1$ polynomials of $n$ variables $P_i(x_1, \ldots, x_n) \in \mathbb{C}[\ x_1, \ldots, x_n]$ do not have common zeroes.

Consider one example. Assume we are interested whether polynomials $P(x,y)$, $Q(x,y)$, $R(x,y)$ have common zeroes.

**Claim.** Let $\texttt{Resultant}[P, Q, x] = S_1(y)$, $\texttt{Resultant}[R, Q, x] = S_2(y)$. If $\texttt{Resultant}[S_1, S_2, y] \neq 0$, then $P$, $Q$, $R$ do not have common zeroes.

**Proof:** Assume there exists common zero $(x_0, y_0)$ of all polynomials $P, Q, R$, then $S_1(y_0) = S_2(y_0) = 0$, hence $\texttt{Resultant}[S_1, S_2, y] = 0$. Contradiction.■

The general construction for $n+1$ polynomials of $n$ variables is exactly the same.

c) On this base explicit center conditions for the equation (5.1) on [0,1] were written in all the cases considered. They turned out to be very simple and transparent, especially in comparison with the equations provided by vanishing of $v_k(1,\lambda)$. See section 5.3. for detailed description of the center set.

## 5.3 Description of a center set for $p$, $q$ of small degrees.

Consider again the polynomial Abel equation:

$$y' = p(x)y^2 + q(x)y^3, \qquad y(0) = y_0$$

with $p(x), q(x)$ – polynomials in $x$ of the degrees $d_1, d_2$ respectively. We will write

$$p(x) = \lambda_{d_1} x^{d_1} + \cdots + \lambda_0,$$

$$q(x) = \mu_{d_2} x^{d_2} + \cdots + \mu_0,$$

$$(\lambda_{d_1}, \ldots, \lambda_0, \mu_{d_1}, \ldots, \mu_0) = (\lambda, \mu) \in \mathbb{C}^{d_1+d_2+2}.$$

Remind that $v_k(x)$ are polynomials in $x$ with the coefficients polynomially depending on the parameters $(\lambda, \mu) \in \mathbb{C}^{d_1+d_2+2}$. Let the **center set** $C \subset \mathbb{C}^{d_1+d_2+2}$ consist of those $(\lambda, \mu)$ for which $y(0) \equiv y(1)$ for all the solutions $y(x)$ of (5.1).

Clearly, $C$ is defined by an infinite number of polynomial equations in $(\lambda, \mu) : v_2(1) = 0, \ldots, v_k(1) = 0, \ldots$ In other words, $C$ is the set of zeroes $Y(I)$ of the ideal $I = \{v_1(1), \ldots, v_k(1), \ldots\}$ in the ring of polynomials $\mathbb{C}[\lambda, \mu]$. In this section we consider $I$ as the ideal in $\mathbb{C}[\lambda, \mu]$ and not in $\mathbb{C}[\ x, \lambda, \mu]$ . We fix one endpoint $a$, say $a = 1$.

The table of multiplicities, obtained in [B1, BY1], gives the number of equation $v_k(1) = 0$, necessary to define $C$ (i.e. the stabilization moment for the set of zeroes of the ideals $I_k(x)$). Since both $v_k(1)$ and $\psi_k(1)$ are polynomials of degree $k-1$ in $(\lambda, \mu)$, the straightforward description of $C$ contains polynomials of a rather high degree, for example up to degree 10 of 9 variables for the case $(\deg P, \deg Q) = (3, 6)$.

The composition conjecture, in contrast, gives us very explicit and transparent equations, describing this center set $C$. Especially explicit are equations in a parametric form (see below).

**5.4.1.** The central set for the equation (5.1) with deg $p =$ deg $q = 2$ has been described in [BFY1]. We remind this result here. Let

$$\begin{aligned} p(x) &= \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 5.3.1** *([BFY1], Theorem V.1) The center set $C \subseteq \mathbb{C}^6$ of the Abel equation (5.1) is given by*

$$\begin{aligned} 2\lambda_2 + 3\lambda_1 + 6\lambda_0 &= 0 \\ 2\mu_2 + 3\mu_1 + 6\mu_0 &= 0 \\ \lambda_2\mu_1 - \lambda_1\mu_2 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$*

**5.4.2** Now let

$$\begin{aligned} p(x) &= \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 5.3.2** *The center set $C \subseteq \mathbb{C}^6$ of the Abel equation (5.1) is given by*

$$\begin{aligned} 2\lambda_2 + 3\lambda_3 &= 0 \\ 2\lambda_1 - \lambda_3 + 4\lambda_0 &= 0 \\ \mu_1 + 2\mu_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$.*

**Proof:** By the composition conjecture, which holds for this case, $p$ and $q$ belong to the center set if and only if $P = \tilde{P}(W)$, $Q = \mu W$ , where $W = x(x-1)$. So, we may assume $Q = \mu x(x-1)$, $P = \alpha W^2 + \beta W$. Thus we get

$$Q = \mu x(x-1) = \frac{\mu_1}{2} x^2 + \mu_0 x$$

$$P = \alpha\left(x(x-1)\right)^2 + \beta x(x-1) = \frac{\lambda_3}{4}x^4 + \frac{\lambda_2}{3}x^3 + \frac{\lambda_1}{2}x^2 + \lambda_0 x$$

Comparing coefficients of $x^k$ in both sides of equalities, we get

$$\begin{cases} \lambda_3 = 4\alpha & \lambda_2 = -6\alpha \\ \lambda_1 = 2\alpha + 2\beta & \lambda_0 = -\beta \\ \mu_1 = 2\mu & \mu_0 = -\mu \end{cases},$$

which is equivalent to the system in the statement of the theorem.■

**5.4.3.** If

$$\begin{aligned} p(x) &= \lambda_1 x + \lambda_0 \\ q(x) &= \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorem one can prove the following

**Theorem 5.3.3** *The center set $C \subseteq \mathbb{C}^6$ of the equation (5.1) is given by*

$$\begin{aligned} 2\mu_2 + 3\mu_3 &= 0 \\ 2\mu_1 - \mu_3 + 4\mu_0 &= 0 \\ \lambda_1 + 2\lambda_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$.*

**5.4.4.** Now let

$$\begin{aligned} p(x) &= \lambda_5 x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 5.3.4** *The center set $C \subseteq \mathbb{C}^8$ of Abel equation (5.1) is given by*

$$\begin{aligned} 5\lambda_5 + 2\lambda_4 &= 0 \\ 10\lambda_5 + 12\lambda_4 + 15\lambda_3 + 20\lambda_2 + 30\lambda_1 + 60\lambda_0 &= 0 \\ \lambda_3 + 4\lambda_2 + 10\lambda_1 + 20\lambda_0 &= 0 \\ \mu_1 + 2\mu_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^8$ is determined by vanishing of the first 8 Taylor coefficients $v_2(1) = 0, \ldots, v_9(1) = 0$.*

**Proof:** By the composition conjecture, which holds for this case, $p$ and $q$ belong to the center set if and only if $P = \tilde{P}(W)$, $Q = \mu W$ , where $W = x(x-1)$. So, we may assume $Q = \mu x(x-1)$, $P = \alpha W^3 + \beta W^2 + \gamma W$. Thus we get

$$\mu x(x-1) = \frac{\mu_1}{2}x^2 + \mu_0 x$$

$$\alpha\left(x(x-1)\right)^2 + \beta\left(x(x-1)\right)^2 + \gamma x(x-1) = \frac{\lambda_5}{6}x^6 + \frac{\lambda_4}{5}x^5 + \frac{\lambda_3}{4}x^4 + \frac{\lambda_2}{3}x^3 + \frac{\lambda_1}{2}x^2 + \lambda_0 x$$

Hence

$$\begin{cases} \lambda_5 = 6\alpha & \lambda_4 = -15\alpha \\ \lambda_3 = 12\alpha + 4\beta & \lambda_2 = -3\alpha - 6\beta \\ \lambda_1 = 2\beta + 2\gamma & \lambda_0 = -\gamma \\ \mu_1 = 2\mu & \mu_0 = -\mu \end{cases},$$

which is equivalent to the system in the statement of the theorem.■

**5.4.5.** If

$$\begin{aligned} p(x) &= \lambda_1 x + \lambda_0 \\ q(x) &= \mu_5 x^5 + \mu_4 x^4 + \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorem one can prove the following

**Theorem 5.3.5** *The center set $C \subseteq \mathbb{C}^8$ of the Abel equation (5.1) is given by*

$$\begin{aligned} 5\mu_5 + 2\mu_4 &= 0 \\ 10\mu_5 + 12\mu_4 + 15\mu_3 + 20\mu_2 + 30\mu_1 + 60\mu_0 &= 0 \\ \mu_3 + 4\mu_2 + 10\mu_1 + 20\mu_0 &= 0 \\ \lambda_1 + 2\lambda_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^8$ is determined by vanishing of the first 4 Taylor coefficients $v_2(1) = 0, \ldots, v_5(1) = 0$.*

**5.4.6.** Now let

$$p(x) = \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0,$$

$$q(x) = \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0.$$

**Theorem 5.3.6** *([BFY2], Theorem 9.2.) The central set $C \subseteq \mathbb{C}^8$ of Abel equation (5.1) consists of two components $C^{(1)}$ and $C^{(2)}$, each of dimension 4.*

*$C^{(1)}$ is given by*

$$\begin{cases} 3\lambda_3 + 4\lambda_2 + 6\lambda_1 + 12\lambda_0 = 0 \\ 3\mu_3 + 4\mu_2 + 6\mu_1 + 12\mu_0 = 0 \end{cases} \tag{5.4.6.1}$$

*and*

$$\begin{cases} \lambda_3\mu_2 - \mu_3\lambda_2 = 0 \\ \lambda_3\mu_1 - \mu_3\lambda_1 = 0 \\ \lambda_2\mu_1 - \mu_2\lambda_1 = 0 \end{cases} \tag{5.4.6.2}$$

*and $C^{(2)}$ is given by* (5.4.6.1) *and*

$$\begin{cases} 3\lambda_3 + 2\lambda_2 = 0 \\ 3\mu_3 + 2\mu_2 = 0 \end{cases} \tag{5.4.6.3}$$

*The set $C$ in $\mathbb{C}^8$ is determined by the vanishing of the first 8 Taylor coefficients $v_2(1) = 0, \ldots, v_9(1) = 0$.*

This theorem was proved in [BFY2] using the fact that the composition conjecture is true for this case. The component (5.4.6.1 & 5.4.6.2) corresponds to the proportionality of $P$ and $Q$, and the component (5.4.6.1 & 5.4.6.3) corresponds to the composition with $W = x(x-1)$.

**5.4.7.** Let

$$\begin{aligned} p(x) &= \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_5 x^5 + \mu_4 x^4 + \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorems one can prove the following

**Theorem 5.3.7** *The center set $C \subseteq \mathbb{C}^9$ of the Abel equation (5.1) is given in a parametric form by*

$$\begin{cases} \lambda_2 = 3\lambda & \lambda_1 = -2\lambda(a+1) \\ \lambda_0 = a\lambda & \mu_5 = 6\alpha \\ \mu_4 = -10\alpha(a+1) & \mu_3 = 4\alpha(a+1)^2 + 8a\alpha \\ \mu_2 = -6\alpha a(a+1) + 3\beta & \mu_1 = 2\alpha a^2 - 2(a+1)\beta \\ \mu_0 = a\beta & \end{cases} \tag{5.4.7.1}$$

*or by*

$$\begin{cases} \mu_4 = -\frac{5\mu_5}{3}\left(\frac{3\lambda_0}{\lambda_2}+1\right) \\ \mu_3 = \frac{2\mu_5}{3}\left(\frac{3\lambda_0}{\lambda_2}+1\right)^2 + 4\frac{\lambda_0}{\lambda_2}\mu_5 \\ \mu_2 = -\frac{3\mu_5\lambda_0}{\lambda_2}\left(\frac{3\lambda_0}{\lambda_2}+1\right) + \frac{\mu_0\lambda_2}{\lambda_0} \\ \mu_1 = \frac{3\mu_5\lambda_0^2}{\lambda_2^2} - 2\left(\frac{3\lambda_0}{\lambda_2}+1\right)\frac{\mu_0\lambda_2}{3\lambda_0} \\ 3\lambda_1 = -2\lambda_2 - 6\lambda_0 \end{cases} \tag{5.4.7.2}$$

*The set $C$ in $\mathbb{C}^9$ is determined by the vanishing of the first 9 Taylor coefficients $v_2(1) = 0, \ldots, v_{10}(1) = 0$.*

**Proof:** We can represent $P = \lambda W$, $Q = \alpha W^2 + \beta W$, where $W = x(x-1)(x-a)$. Thus

$$\lambda\left(x^3 - (a+1)x^2 + ax\right) = \frac{\lambda_2}{3}x^3 + \frac{\lambda_1}{2}x^2 + \lambda_0 x,$$

$$\alpha\left(x^6 + (a+1)^2x^4 + a^2x^2 - 2(a+1)x^5 + 2ax^4 - 2a(a+1)x^3\right) +$$
$$\beta\left(x^3 - x^2(a+1) + ax\right) = \frac{\mu_5}{6}x^6 + \ldots + \mu_0 x$$

Comparing coefficients by $x^k$ in both sides of equalities, we obtain (5.4.7.1). After some transformations we obtain (5.4.7.2). (Notice, that $\lambda_2 \neq 0$ as leading coefficient.) ■

## 5.4 Composition conjecture for some special families of polynomials

In this section we generalize M.Sc. thesis results, finding out some classes of polynomials $p$, $q$ of arbitrary high degree, for which condition for a center on a given set of numbers can be explicitly found (and coincide with composition case).

Let $a_1 = 0, a_2, \dots, a_\ell$ be given. Consider any polynomial $W(x)$ vanishing at all the points $a_j, \quad j = 1, \dots, \ell$.

**Theorem 5.2.2** *Assume that for at least one* $a_j$, $\int_0^{a_j} W^k dx \neq 0$ *and* $\int_0^{a_j} W^n dx \neq 0$. *Polynomials* $p = W^k(\alpha + \beta W')$, $q = W^n(\gamma + \delta W')$ *define center on* $[0; a_1; \dots; a_\ell]$ *if and only if* $\alpha = \gamma = 0$.

**Remark:** Notice, that the condition "$\int_0^{a_j} W^k dx \neq 0$ for at least one $a_j$" is satisfied, for instance, for $W(x) = \prod_{i=1}^{\ell}(x - a_i)$, where all $a_j$ are different. Indeed, consider the function $f(x) = \int_0^x W(t)^k dt$. If all $a_j, j = 1, \dots, \ell$ would be zeroes of $f(x)$, then $\deg f \geq (k+1)\ell$. But $\deg W = \ell$, so $\deg f(x) = k\ell + 1$. We obtain $k\ell + 1 \geq (k+1)\ell$, which is not satisfied for $\ell > 1$ .

Similarly one can show that $W(x) = \prod_{i=1}^{\ell}(x - a_i)^{m_i}$ satisfies the condition "$\int_0^{a_j} W^k dx \neq 0$ for at least one $a_j$" for almost all $k$, and so on. So, this condition is "almost generic".

**Proof:** Since $\psi_2(x) = P(x)$, the conditions $\psi_2(a_j) = 0$ imply $\alpha = 0$. Since $\psi_3(x) = P^2(x) - Q(x)$, the conditions $\psi_3(a_j) = 0$ imply $\gamma = 0$. ■

**Theorem 5.2.3** *Assume that* $\deg W > 2$ *and for at least one* $a_j$

$$\det \begin{vmatrix} \int_0^{a_j} W^n dx & \int_0^{a_j} W^n W'' dx \\ \int_0^{a_j} W^{n+k+1} dx & \int_0^{a_j} W^{n+k+1} W'' dx \end{vmatrix} \neq 0.$$

*Polynomials* $p = W^k(\alpha + \beta W')$, $q = W^n(\gamma + \delta W' + \epsilon W'')$ *define center on* $[0; a_1; \dots; a_\ell]$ *if and only if* $\alpha = \gamma = \epsilon = 0$.

**Proof:** The conditions $\psi_2(a_j) = 0$ imply $\alpha = 0$. The conditions $\psi_3(a_j) =$

0 imply

$$\gamma \int_0^{a_j} W^n + \epsilon \int_0^{a_j} W^n W'' = 0,$$

and the conditions $\psi_4(a_j) = 0$ imply

$$\gamma \int_0^{a_j} W^{n+k+1} + \epsilon \int_0^{a_j} W^{n+k+1} W'' = 0.$$

If the determinant of the system is nonzero, we get that the system has the only zero solution. ■

# Chapter 6

# Riemann Surface approach to the Center problem for Abel equation

## 6.1 Introduction

In this chapter Riemann Surface approach to the Center problem for Abel equation is discussed. This is a convenient general setting, where Abel Equation

$$\frac{dy}{dz} = y^2 \frac{dP(z)}{dz} + y^3 \frac{dQ(z)}{dz}$$

considered on a given Riemann Surface. In this chapter

- we generalize notions of center and composition to the case of Riemann surfaces;
- following Chapter 3, we show that all the facts about Poincare return map and recurrence relations to compute its coefficients remain valid in this setting.

## 6.2 Abel equation on Riemann Surfaces

Let $X$ be a domain on a connected Riemann Surface and let $P$ and $Q$ be two analytic functions on $X$. For Abel equation related to planar vector fields $X$

is a neighborhood of the unit circle $S^1$ on $\mathbb{C}$, Laurent polynomials $P$ and $Q$ are analytic on $X$. We consider the following **Abel differential equation** on $X$:

$$dy = y^2\,dP + y^3\,dQ \tag{A}$$

A (local) solution of (A) is an analytic function $y$ on an open set $\Omega$ in $X$, such that the differential forms $dy$ and $y^2\,dP + y^3\,dQ$ coincide in $\Omega$.

If $x$ is a local coordinate in $\Omega$, (A) takes the usual form

$$\frac{dy}{dx} = y^2\,p(x) + y^3\,q(x),$$

where

$$p(x) = \frac{d}{dx}P(x),\ Q(x) = \frac{d}{dx}Q(x).$$

Let $Y \to X$ be the universal covering of $X$. The equation (A) can be lifted onto $Y$. One can easily show, that for any $a \in Y$ and for any $c \in \mathbb{C}$, there is a unique solution $y_c$ of (A) on $Y$, satisfying $y_c(a) = c$, whose singularities tend to infinity as $c$ tends to zero. In what follows we always assume that $c$ is sufficiently small, so $y_c$ is regular and univalued on any compact part of $Y$, but in general is multivalued on $X$.

**Definition 6.2.1** *Let $\gamma$ be a closed curve in $X$. We say that the Abel equation (A) has* **a center along the curve** $\gamma$*, if for any small $c \in \mathbb{C}$ $y_c$ is univalued along $\gamma$.*

Notice that in this definition it is sufficient to assume that for any sufficiently small $c$ and some $a \in \gamma$, $y_c(a) = y_{c,\gamma}(a)$, where $y_{c,\gamma}$ is a result of an analytic continuation of $y_c$ along $\gamma$. Indeed, by uniqueness of a solution of the first order differential equation (A), $y_c(a) = y_{c,\gamma}(a)$ implies that $y_c$ and $y_{c,\gamma}$ coincide in a neighborhood of $a$. Then by analytic continuation $y_c$ coincide with $y_{c,\gamma}$ along the the whole $\gamma$, i.e. $y_c$ is univalued.

**Definition 6.2.2** *We say that (A) has a* **total center on** $X$*, if it has a center along any closed curve in $X$.*

In particular, this is always the case for $X$ – simply-connected. Indeed, in this case any closed curve is homotopic to a point, so after analytic continuation along any closed curve we will obtain the initial value at this point.

**Definition 6.2.3** *Let $\tilde{X}$ be a domain on another Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ be analytic functions on $\tilde{X}$. Assume there is an analytic mapping $w : X \to \tilde{X}$, such that $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$. We say that the Abel equation (A) on $X$* **is induced from** *the Abel equation*

$$dy = y^2 \, d\tilde{P} + y^3 \, d\tilde{Q} \tag{$\tilde{A}$}$$

*on $\tilde{X}$ by the mapping $w$. We also say that (A)* **is factorized through** *$\tilde{X}$.*

Notice, that the words "factorized through $w : X \to \tilde{X}$" are equivalent to the words "composition representation $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$.". When we deal with Abel differential equation, we shall use the word "factorization", and when we deal with $P$ and $Q$, we shall use the word "composition".

**Lemma 6.2.1** *Let (A) be induced from ($\tilde{A}$) by $w$. Then any solution $y$ of (A) is induced from a corresponding solution $\tilde{y}$ of ($\tilde{A}$), i.e. $y = \tilde{y} \circ w$.*

**Proof:** Let $y = y_c$ take a value $c$ at some point $a \in X$. Consider the solution $\tilde{y}_c$ of ($\tilde{A}$), taking the value $c$ at $w(a) \in \tilde{X}$. By the "invariance of the first differential", $\tilde{y}_c \circ w$ is a solution of ($A$), and it satisfies $\tilde{y}_c \circ w(a) = c$. Hence locally, $\tilde{y}_c \circ w \equiv y_c$, and analytic continuation completes the proof. ■

**Corollary 6.2.1** *If (A) is induced from ($\tilde{A}$) by $w$, and if ($\tilde{A}$) has a center along $w(\gamma)$, then (A) has a center along $\gamma$.*

**Definition 6.2.4** *Let $a$ and $b$ be two different points in $X$. We say that $a$ and $b$ are* **conjugate with respect to the equation (A) and with respect to a certain homotopy class of curves $\gamma$, joining $a$ and $b$ in** *$X$, if for any solution $y_c$ of (A) with sufficiently small $c$ ($y_c(a) = c$), its continuation $y_{c,\gamma}$ along $\gamma$ satisfies $y_{c,\gamma}(b) = c$. In other words, any solution of (A) takes equal values at $a$ and $b$ after analytic continuation along $\gamma$.*

A priori it is not evident that these definitions are natural and that conjugate points can appear at all. However, the following proposition gives a basic reason for their appearance:

**Proposition 6.2.1** *Let $X$, $\tilde{X}$, $P$, $Q$, $\tilde{P}$, $\tilde{Q}$, $w$ be as above. Consider two different points $a$, $b \in X$ and a path $\gamma$ joining them. If $w(a) = w(b)$ and $(\tilde{A})$ has a center along the closed curve $w(\gamma)$, then $a$ and $b$ are conjugate along $\gamma$ for the Abel equation (A). In particular, if $\tilde{X}$ is simply connected, any two points $a$, $b$ with $w(a) = w(b)$ are conjugate along any $\gamma$ joining them.*

**Proof:** Follows immediately from the last lemma. ■

### 6.2.1 Example: Polynomial Composition Conjecture in the case $X = \mathbb{C}$, $P$ and $Q$ – polynomials in $x$

Here $X$ is simply-connected, $P$ and $Q$ are analytic on $X$ and hence the Abel equation

$$\frac{dy}{dx} = p(x)\, y^2 + q(x)\, y^3, \tag{6.1}$$

with $p(x) = \dfrac{d}{dx}P(x)$, $q(x) = \dfrac{d}{dx}Q(x)$, has a center along any closed curve $\gamma$.

As far as a factorization of (6.1) is concerned, assume that there exist polynomials $w$, $\tilde{P}$, $\tilde{Q}$, such that $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$. Then for $\tilde{X} = \mathbb{C}$ (6.1) is induced by $w$ from

$$\frac{dy}{dw} = \tilde{P}'(w)\, y^2 + \tilde{Q}'(w)\, y^3. \tag{6.2}$$

By proposition 5.2.1, any two points $a$ and $b$ such that $w(a) = w(b)$ are conjugate along any path $\gamma$.

In [BFY1-4] and [BY] this example was investigated in some details. In particular, for small degrees of $P$ and $Q$ it was shown, that conjugate points can appear only in this way.

The following conjecture was proposed in [BFY1]:

**Polynomial Composition Conjecture:** *Two different points $a$ and $b$ in $\mathbb{C}$ are conjugate for the Abel equation (A) with $P$, $Q$ - polynomials in $x$, if and only if the following* **Polynomial Composition Condition** *is satisfied: there exists a factorization $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$ with polynomial mapping $w$, such that $w(a) = w(b)$.*

Notice, that although the equation (A) is non-symmetric with respect to $p$ and $q$, this conjecture proposes the symmetric condition: if $a$ and $b$ are conjugate for (A), then they are conjugate also for the equation

$$\frac{dy}{dx} = q(x)\, y^2 + p(x)\, y^3. \tag{6.3}$$

This situation is not unique in the center problem. In [Che2] it was shown that the Lienard system

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x) = 0$$

has a center at the origin if and only if the functions $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$ can be represented as a composition $F(x) = \tilde{F}(z(x))$, $G(x) = \tilde{G}(z(x))$ for an analytic function $z(x)$, with $z'(0) < 0$.

### 6.2.2 Example: Laurent Composition Condition in the case $X$ – a neighborhood of the unit circle $S^1 = \{|x| = 1\} \subseteq \mathbb{C}$, $P$ and $Q$ – Laurent series, convergent on $X$

We may rewrite Abel differential equation in an invariant form

$$d\rho = dP(\theta)\, \rho^2 + dQ(\theta)\, \rho^3,$$

then expressing $\sin\theta$ and $\cos\theta$ through $x = e^{i\theta}$, i.e.

$$\begin{cases} \cos\theta = \frac{x+x^{-1}}{2}, \\ \sin\theta = \frac{x-x^{-1}}{2i}, \\ \rho = y, \end{cases}$$

we obtain $P$ and $Q$ in the form of **Laurent polynomials** in $x$, and the differential equation is

$$dy = y^2\, dP + y^3\, dQ$$

considered on the circle $x([0, 2\pi]) = S^1$.

We shall discuss this case in much more detail in the next chapter, because it corresponds directly to the classical Center-Focus Problem for homogeneous polynomial vector fields on the plane.

The one of possible factorizations in this case, which we shall call **Laurent Composition Condition**, takes the form $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$, where $w$ is a Laurent series, and $\tilde{P}$, $\tilde{Q}$ are regular analytic functions on the disk $D$ in $\mathbb{C}$, containing the image $w(S^1)$.

**Lemma 6.2.2** *Laurent Composition Condition implies center along* $S^1$.

**Proof:** We have a factorization $w : X \to D$, and since $D$ is simply connected, the Abel equation $dy = y^2 d\tilde{P} + y^3 d\tilde{Q}$ on $D$ has a total center, then by Corollary 5.2.1 it implies center along $S^1$. ■

In this case the conjecture that this is the only reason for center is not true: there are known cases of center along $S^1$ for the equation (A), when $p$ and $q$ can not be represented as a Laurent composition (the example was considered by Alwash in [A] – see Chapter 2 above).

Nevertheless, for $p(z)$, $q(z)$ – Laurent polynomials of small degrees up to (4,4), i.e. of the form $z^{-2}(a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0)$, the composition representation is the only possible reason for center (shown in Chapter 4).

## 6.3 Poincare mapping

Here we discuss notions introduced in Chapter 3, in a more general setting. Let us return to a general situation of the Abel equation (A)

$$dy = y^2\, dP + y^3\, dQ$$

on a Riemann Surface $X$. Let $a, b \in X$ and let $\gamma$ be a curve in $X$, joining $a$ and $b$.

Let us consider a general solution $y_{c,\gamma}$ of the equation (A) along $\gamma$, taking the value $c$ at the point $a$: $y_{c,\gamma}(a) = c$. For sufficiently small $c$ we may represent it in a form of a (convergent) power series in $c$:

$$y_{c,\gamma}(x) = c + \sum_{k=2}^{\infty} v_k\, c^k, \qquad (6.1)$$

where $v_k = v_k(a, x, \gamma)$ are (multivalued) functions on $X$, $x$ is a point on $\gamma$:

For fixed $a$, $\gamma$ substituting the solution $y_{c,\gamma}(x)$ in the form of expansion $y_{c,\gamma}(x) = c + \sum_{k=2}^{\infty} v_k(x)\, c^k$ into (A), we obtain the following recurrence relation on $v_k(x)$:

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ \text{ and for } n \geq 2 \\ v_n(a) = 0 \quad \text{ and} \\ dv_n(x) = dP(x) \displaystyle\sum_{i+j=n} v_i(x)v_j(x) + dQ(x) \sum_{i+j+k=n} v_i(x)v_j(x)v_k(x) \end{cases} \tag{6.2}$$

**Definition 6.3.1** *The* **Poincare mapping** $\Psi_\gamma : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *of the equation (A) along* $\gamma$ *is defined by*

$$\Psi_\gamma(c) = y_{c,\gamma}(b), \tag{6.3}$$

*where* $y_{c,\gamma}$ *is, as above, the solution of (A), taking the value* $c$ *at the point* $a$*, and analytically continued along* $\gamma$ *up to the point* $b$*.*

As it was shown,

$$\Psi_\gamma(c) = c + \sum_{k=2}^{\infty} v_k(b)\, c^k \tag{6.4}$$

For $X = \mathbb{C}$, $a = 0$, $b = 1$, we obtain the usual Poincare mapping as we used in [BY].

**Lemma 6.3.1** *1) The equation (A) has a center along unclosed curve* $\gamma$ *with end points* $a$ *and* $b$ *(i.e. points* $a$ *and* $b$ *are conjugate) if and only if* $v_k(b) = 0$ *for any* $k \geq 2$*. Here* $v_k(x)$ *(for fixed* $a$, $\gamma$*) are obtained by integration along* $\gamma$ *of (6.2).*
*2) The equation (A) has a center along closed curve* $\gamma$ *if and only if* $v_k(x)$ *are univalued functions along* $\gamma$*.*

**Proof:** Follows immediately from definitions of a center and Poincare mapping. ■

The recurrence relations (6.2) can be in fact linearized in the following sense: Let us fix the curve (and the end-points $a$ and $b$) and consider an inverse function (with respect to $y$ and $c$) to $y_{c,\gamma}(x)$:

$$c = y^{-1}(x, y_{c,\gamma}(x)) = y^{-1}(x, y)$$

One can easily see that for sufficiently small $c$ and $y$ we can represent $y^{-1}$ by a convergent power series

$$c = y^{-1}(x, y) = y + \sum_{k=2}^{\infty} \varphi_k(x) y^k, \tag{6.5}$$

where $\varphi_k(a) = 0$, since $y^{-1}(a, c) = c$.

**Lemma 6.3.2 (BFY1)** *The coefficients $\varphi_k(x)$ satisfy the following recurrence relation:*

$$\begin{cases} \varphi_0(x) \equiv 0,\ \varphi_1(x) \equiv 1 \ \textit{and for}\ \ k \geq 2 \\ d\varphi_k(x) = -(k-1)dP(x)\varphi_{k-1}(x) - (k-2)dQ(x)\varphi_{k-2}(x) \\ \varphi_k(a) = 0 \end{cases} \tag{6.6}$$

**Proof:** Write (6.5) as

$$c = \sum_{k=0}^{\infty} \varphi_k(x) y^k,\ \varphi_0 \equiv 0,\ \varphi_1 \equiv 1 \tag{6.7}$$

Taking a differential of it with respect to $x$ along the solution of (A), we get

$$0 = \sum_{k=0}^{\infty} d\varphi_k(x) y^k + \sum_{k=0}^{\infty} k\varphi_k(x) y^{k-1} \cdot dy = \sum_{k=0}^{\infty} d\varphi_k y^k +$$

$$\sum_{k=2}^{\infty} ((k-1)dP(x)\varphi_{k-1}(x) + (k-2)dQ(x)\varphi_{k-2}(x)) y^k$$

Equating to zero the right-hand side coefficients produces the required recurrence. ■

**Definition 6.3.2** *The* **Inverse Poincare mapping** $\Psi_\gamma^{-1} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *of the equation (A) along* $\gamma$ *is defined by*

$$\Psi_\gamma^{-1}(c) = y_{\tilde{c},\gamma}(b), \tag{6.8}$$

*where* $y_{\tilde{c},\gamma}$ *is, as above, the solution of (A), taking the value* $\tilde{c}$ *at the point* $a$*, and* $\tilde{c}$ *is chosen such that* $y_{c,\gamma}(b) = \tilde{c}$*. Shortly,* $\Psi \circ \Psi^{-1} \equiv id$*.*

As it was shown,

$$\Psi^{-1}(\tilde{c}) = \tilde{c} + \sum_{k=2}^{\infty} \varphi_k(b)\, \tilde{c} \tag{6.9}$$

Notice that in our considerations we were free to choose the endpoint $b$. It leads immediately to the following

**Lemma 6.3.3 (BFY1)** *For any* $q \geq 2$*, for any* $b \in \mathbb{C}$ *the ideals* $I_q = \{v_2(b), \ldots, v_q(b)\}$ *and* $\hat{I}_q = \{\varphi_2(b), \ldots, \varphi_k(b)\}$ *coincide.*

**Proof:** This follows immediately from the standard expressions for the Taylor coefficients of the inverse function: if $y = x + a_2x^2 + a_3x^3 + \cdots$, $x = y + b_2y^2 + b_3y^3 + \cdots$, then $b_2 = -a_2$, $b_3 = -a_3 + 2a_2^2, \ldots$, $a_2 = -b_2$, $a_3 = -b_3 + 2b_2^2, \ldots$ ■

Now we get as the corollary the following theorems:

**Theorem 6.3.1** *The equation (A) has a center along an unclosed curve* $\gamma$ *with end points* $a$ *and* $b$ *(i.e. points* $a$ *and* $b$ *are conjugate) if and only if* $\varphi_k(b) = 0$ *for any* $k \geq 2$*. Here* $\varphi_k(x)$ *(for fixed* $a$*,* $\gamma$*) are obtained by integration along* $\gamma$ *of the linear recurrence (6.6).*

**Theorem 6.3.2** *The equation (A) has a center along a closed curve* $\gamma$ *if and only if all the functions* $\varphi_k(x)$ *are univalued functions along* $\gamma$*.*

# Chapter 7

# Factorization (composition) for the case $X \subseteq \mathbb{C}$, $P$ and $Q$ – rational functions

## 7.1 Introduction

We started from the center problem for vector fields on the plane. It was reduced to the center problem for Abel differential equation with $p$, $q$ – Laurent polynomials. The natural question is to study Abel equation with $p$, $q$ – rational functions. Everything, what can be said in general about rational functions, remains valid for the case of Laurent polynomials.

It turns out that the assumption that $P$ and $Q$ are rational functions puts the composition (factorization) question completely in the framework of algebraic geometry of rational curves and of the Ritt theory ([R1]). We show that the examples of Polynomial composition and Laurent composition, given in sections 6.2.1 and 6.2.2, are in some sense "generic", and any analytic factorization of rational function can be reduced to composition of rational functions.

## 7.2 Composition and rational curves

The following facts are very basic in algebraic geometry. We restate them for convenience of our presentation. For details we address the reader to any classical algebraic geometry text (e.g. [Sha]).

We'll denote by $Y$ the curve in $\mathbb{C}^2$ , parameterized by $P$, $Q$:
$Y = \{(P(t), Q(t)), t \in \mathbb{C}\}$.

**Lemma 7.2.1** *The curve $Y$ for* **rational** *$P$ and $Q$ is an algebraic curve.*

**Proof:** We need to prove that for sufficiently large $d$ there exists a polynomial $F(x,y)$ of the degree $d$, such that $F(P(t),Q(t)) \equiv 0$. Without loss of generality we may assume that $P$ and $Q$ have the same denominator $R$. Consider all the products $P(t)^i Q(t)^j R^d$, $i+j \le d$. These are polynomials of $t$ of the degrees less than or equal to $i \deg P + j \deg Q + d \deg R \le d(\deg P + \deg Q + \deg R)$. But there exist $\frac{d(d-1)}{2}$ such products $P^i Q^j$, therefore for $\frac{d(d-1)}{2} > d(\deg P + \deg Q + \deg R)$, i.e. for $d > 2(\deg P + \deg Q + \deg R) + 1$ there exists a linear dependence over $\mathbb{C}$:

$$\sum_{i,j} \beta_{ij} P(t)^i Q(t)^j R(t)^d \equiv 0, \text{hence } \sum_{i,j} \beta_{ij} P(t)^i Q(t)^j \equiv 0.$$

This polynomial $F(x,y) = \sum \beta_{ij} x^i y^j$ vanishes on $Y$. Such polynomial of minimal degree defines an algebraic curve $Y$. ■

**Lüroth theorem.** *Any subfield of a field of rational functions is generated by a rational function.*

**Corollary 7.2.1** *There exist rational functions $r(t)$, $\bar{P}$, $\bar{Q}$, s.t. $P(t) = \bar{P}(r(t))$, $Q(t) = \bar{Q}(r(t))$ and s.t. the map*

$$\bar{\gamma} : \mathbb{C} \to Y, \quad z \mapsto (\bar{P}(z), \bar{Q}(z))$$

*defines a birational isomorphism between $\mathbb{C}$ and $Y$. In particular, $Y$ is a rational curve.*

**Proof:**
**1.** Notice that $K = \mathbb{C}(P(t), Q(t))$ is is a subfield of the field of rational functions $\mathbb{C}(t)$. By Lüroth theorem $K = \mathbb{C}(r(t))$ for some rational function $r(t)$. In particular, $P$ and $Q$ belong to $K$, hence there exist rational functions $\bar{P}(t)$, $\bar{Q}(t)$ such that $P(t) = \bar{P}(r(t))$, $Q(t) = \bar{Q}(r(t))$

**2.** It is obvious that $\bar{\gamma}$ is surjective and rational. Let's prove that there exists an inverse map $\bar{\gamma}^{-1} : Y \to \mathbb{C}$. Since $r(t) \in \mathbb{C}(P(t), Q(t))$, there exist a rational function $R(x, y)$ s.t. $R(P(t), Q(t)) \equiv r(t)$, i.e. $R(\bar{P}(r(t)), \bar{Q}(r(t))) \equiv r(t)$. Obviously it is a rational function from $Y$ to $\mathbb{C}$. Let us prove that $R = \bar{\gamma}^{-1}$. Indeed, $R \circ \bar{\gamma} \equiv id : \mathbb{C} \to \mathbb{C}$: $R(\bar{\gamma}(z)) = R(\bar{P}(z), \bar{Q}(z)) = R(P(r(t)), Q(r(t))) = r(t) = z$, since for any $z$ there exists $t$: $z = r(t)$. Vice versa: $\bar{\gamma} \circ R \equiv id : Y \to Y$, since $\bar{\gamma}(R(P(t), Q(t))) = \bar{\gamma}(r(t)) = (\bar{P}(r(t)), \bar{Q}(r(t))) = (P(t), Q(t))$. ■

**Definition 7.2.1** *The* **degree of a map** $\gamma = (P, Q) : \mathbb{C} \to Y$ *is the degree of the algebraic extension* $[\mathbb{C}(t) : \mathbb{C}(P, Q)]$.

**Definition 7.2.2** *The parameterization of a rational curve* $Y$, $\gamma : \mathbb{C} \to Y$ $z \mapsto (P(z), Q(z))$ *is called* **minimal** *if* $\deg \gamma = 1$.

From corollary 5.3.1 it follows that a minimal parameterization defines a birational isomorphism between $\mathbb{C}$ and $Y$.

**Definition 7.2.3** *The mapping* $f : X \to Y$ *is called* **"not 1-1"**, *if there exists an open set* $\Omega \subseteq Y$, *s.t. each point of* $\Omega$ *has more than one preimage under* $f$.

**Definition 7.2.4** *A rational function* $r$ *is called* **common divisor under composition** *of rational functions* $P$ *and* $Q$, *if* $P = \tilde{P}(r)$, $Q = \tilde{Q}(r)$. *The common divisor* $r$ *is called* **nontrivial**, *if* $r$ *is not 1-1.*

**Definition 7.2.5** *A rational function* $r$ *is called* **Composition Greatest Common Divisor (CGCD)** *of rational functions* $P, Q$, *if* $r$ *is a common divisor under composition of* $P$ *and* $Q$, *and if* $\tilde{r}$ *is another common divisor of* $P$ *and* $Q$ *under composition, then* $r = R(\tilde{r})$ *for a rational function* $R$.

**Definition 7.2.6** *The* **degree of a rational function** *is a maximum of degrees of numerator and denominator.*

Let's notice that among rational functions only linear functions (functions of degree 1) are 1-1. Respectively, if we are looking for nontrivial CGCD in the class of rational functions, it must have degree greater than 1. It is easy to show that CGCD exists and satisfies all the properties of usual greatest common divisor. In particular,

**Proposition 7.2.1** *For any rational functions $P(t)$, $Q(t)$ their CGCD $r(t)$ exists and is given by corollary 5.3.1. CGCD is unique in the algebra of rational functions under compositions up to composition with an invertible rational function (i.e. a function of degree 1), i.e. two CGCD of a given function can be obtained each one from another by a (right or left) composition with a linear function.*

**Proof:**
Obviously, $r(t)$ from Corollary 5.3.1 defines rational common divisor under composition. If $\tilde{r}$ is another composition common divisor of $P$ and $Q$, then $P = \tilde{P}(\tilde{r})$, $Q = \tilde{Q}(\tilde{r})$, so $\mathbb{C}(\tilde{r}(t)) \supseteq \mathbb{C}[P(t), Q(t)] = \mathbb{C}(r(t))$, hence $r(t) = R(\tilde{r}(t))$ for some rational function $R$. Therefor $r(t)$ is actually a CGCD.

If $r$ and $\tilde{r}$ are two CGCD, then $r(t) = R(\tilde{r}(t))$, but $\tilde{r}(t) = \tilde{R}(r(t))$, so $R \circ \tilde{R} = id$, hence $R$ is a linear function. ■

The following facts are proved, for example, in [Sha]:

**Lemma 7.2.2** *1) For a rational map $\gamma = (P, Q) : \mathbb{C} \to Y$ the number of preimages of almost each point is equal to* $\deg \gamma$.
*2)* $[\mathbb{C}(t) : \mathbb{C}(r(t))] = \deg r(t)$.

**Corollary 7.2.2** *The degree of the map $\gamma = [P, Q]$ is equal to the degree of the rational CGCD of $P$ and $Q$. If $r$ is CGCD of $P$ and $Q$, $P = \bar{P}(r)$, $Q = \bar{Q}(r)$, then $\bar{\gamma} : \mathbb{C} \to Y$ $z \mapsto (\bar{P}(z), \bar{Q}(z))$ is a minimal parameterization.*

The following two results show that allowing an analytic (and not a priori rational) composition does not add, in fact, anything new.

**Lemma 7.2.3** $\deg \gamma > 1$ *if and only if there exists an analytic factorization $w : \mathbb{C} \to \tilde{X}$, where $\tilde{X}$ is a Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ are analytic functions on $\tilde{X}$, such that $P(t) = \tilde{P}(w(t))$, $Q(t) = \tilde{Q}(w(t))$, and $w$ is not 1-1.*

**Proof:**
Let $\deg \gamma > 1$, then taking $w = r : \mathbb{C} \to \mathbb{C}$ we get the required analytic factorization. Vice versa, let there exist an analytic factorization $w : \mathbb{C} \to \tilde{X}$, $P(t) = \tilde{P}(w(t))$, $Q(t) = \tilde{Q}(w(t))$. Then $\mathbb{C}(P, Q) = \mathbb{C}(\tilde{P}(w), \tilde{Q}(w))$, which is a proper subset in $\mathbb{C}(t)$, because $w(t)$ glues some points in $\mathbb{C}$, but in $\mathbb{C}(t)$ there are functions which map these points into different points. Hence $[\mathbb{C}(t) : \mathbb{C}(P, Q)] > 1$. ■

**Corollary 7.2.3** *If there exists an analytic factorization of rational functions $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$ with $w$ – not 1-1, then there exists a nontrivial CGCD of $P$ and $Q$: $P = \bar{P}(r)$, $Q = \bar{Q}(r)$ for $\bar{r}$ of the degree greater than 1.*

## 7.3 Structure of composition in the case $X = \mathbb{C}$, $P$ and $Q$ – polynomials

We shall prove that the factorization in the form of **Polynomial Composition Condition** is essentially the only one natural factorization in the polynomial case, namely:

**Theorem 7.3.1** *Assume there exists an analytic factorization $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$ with $w$ – not 1-1. Then there exists a polynomial factorization $P(t) = \hat{P}(\hat{w}(t))$, $Q(t) = \hat{Q}(\hat{w}(t))$, with $\hat{P}$, $\hat{Q}$, $\hat{w}$ being polynomials, the degree of $\hat{w}$ is greater than 1 and* $\deg(\hat{P}, \hat{Q}) = 1$.

The proof will follow from the lemma:

**Lemma 7.3.1** *If we have a factorization $P = \bar{P} \circ r$, $Q = \bar{Q} \circ r$ with $\bar{P}(t)$, $\bar{Q}(t)$, $r(t)$ - rational functions: $\mathbb{C} \to \mathbb{C}$, then there exists a linear rational function $\lambda : \mathbb{C} \to \mathbb{C}$, s.t. $\bar{P} \circ \lambda$, $\bar{Q} \circ \lambda$, $\lambda^{-1} \circ r$ are polynomials.*

**Proof:**
This proof is contained essentially in [R]. Let $r(\infty) = a$. Let the degrees of the rational functions $r$, $\bar{P}$ be $n$, $m$ respectively. The degree of the polynomial $P$ will be $mn$. Then:
$r(\infty) = a$ with multiplicity not more than $n$, $\bar{P}(a) = \infty$ with multiplicity not more than $m$, but $\bar{P} \circ r(\infty) = \infty$ with multiplicity exactly $mn$, because $\bar{P} \circ r$ is a polynomial.

Hence we get that $r(\infty) = a$ with multiplicity $n$, $\bar{P}(a) = \infty$ with multiplicity $m$.

Now take $\lambda(z) = \frac{1}{z-a}$, i.e. $\lambda^{-1}(a) = \infty$. Then $\bar{P} \circ \lambda(\infty) = \infty$ with multiplicity $n$, $\lambda^{-1} \circ r(\infty) = \infty$ with multiplicity $m$, hence they are polynomials. Similarly $\bar{Q} \circ \lambda$ is a polynomial. ■

**Proof of theorem:**
By corollary 5.3.3 there exists a factorization $P(t) = \bar{P}(r(t)$, $Q(t) = \bar{Q}(r(t)$,

for some rational functions $\bar{P}(t)$, $\bar{Q}(t)$ and rational function $r(t)$ of degree greater than 1, such that $\deg(\bar{P},\bar{Q})=1$. Then taking $\hat{P}=\bar{P}\circ\lambda$, $\hat{Q}=\bar{Q}\circ\lambda$, $\hat{w}=\lambda^{-1}\circ r$ we obtain the required polynomial factorization $P(t)=\hat{P}(\hat{w}(t))$, $Q(t)=\hat{Q}(\hat{w}(t))$ with $\hat{w}$ of degree greater than 1 and $\deg(\hat{P},\hat{Q})=1$. ■

The similar fact was proved by C. Christopher in [Chr], when he investigated polynomial case of Lienard system.

**Corollary 7.3.1** *If* $\deg[P,Q]=s>1$, *then the two polynomials* $P$ *and* $Q$ *have a nontrivial CGCD of the degree* $s$ *in the algebra of polynomials under composition:* $P(t)=\hat{P}(r(t))$, $Q(t)=\hat{Q}(r(t))$, *with* $\hat{P}$, $\hat{Q}$, $r$ *– algebraic polynomials and* $\deg r=s$. *The map* $\gamma:\ z\mapsto(\hat{P}(z),\hat{Q}(z))$ *defines a minimal polynomial parameterization of the algebraic curve* $Y=\{(P(t),Q(t)),t\in\mathbb{C}\}$.

## 7.4 Structure of composition in the case $X$ – a neighborhood of the unit circle on $\mathbb{C}$, $P$ and $Q$ – Laurent polynomials

We describe possible factorization of Laurent polynomials, up to a natural equivalence.

**Definition 7.4.1** *The composition representations* $P=\bar{P}(\bar{r})$ *and* $P=\tilde{P}(\tilde{r})$ *are* **equivalent** *if there exists a linear rational function* $\lambda$, *s.t.* $\bar{P}=\tilde{P}(\lambda)$, $\bar{r}=\lambda^{-1}(\tilde{r})$.

**Theorem 7.4.1** *Up to the equivalence relation of definition 3.8 there are only two types of composition representations of a Laurent polynomial* $P$*:*
*(1)* $P=\bar{P}(r)$, *where* $\bar{P}$ *is a usual (algebraic) polynomial, and* $r$ *is a Laurent polynomial.*
*(2)* $P=\bar{P}(r)$, *where* $\bar{P}$ *is a Laurent polynomial, and* $r=z^k$ *for some* $k\in\mathbb{N}$, $k\geq 2$.
*Any two composition representations of types (1) and (2) for* $\deg r>1$ *and* $k>1$ *are not equivalent.*

**Proof:**

$P$ is a Laurent polynomial, hence $\infty$ has exactly two preimages – 0 and $\infty$. Assume we are given a composition in the class of rational functions: $P = \tilde{P}(\tilde{r})$ . We shall show that by choosing a suitable linear rational function $\lambda$ we obtain $P = (\tilde{P} \circ \lambda) \circ (\lambda^{-1} \circ \tilde{r})$, where $\bar{P} = \tilde{P} \circ \lambda$ and $r = \lambda^{-1} \circ \tilde{r}$ are of the required form (1) or (2).

$\tilde{P}$ may have either two or one preimage of $\infty$.

1) Assume first that $\infty$ has two preimages $a \neq b$ under the map $\tilde{P}$: $\tilde{P}(a) = \tilde{P}(b) = \infty$. Take a linear function $\lambda(z)$, s.t. $\lambda(0) = a$, $\lambda(\infty) = b$: $\lambda(z) = \dfrac{1}{z + \frac{1}{a-b}} + b$. Then $(\tilde{P} \circ \lambda)(0) = \infty$, $\quad (\tilde{P} \circ \lambda)(\infty) = \infty$, so $\bar{P} = \tilde{P}(\lambda)$ is a Laurent polynomial. Then necessary $(\lambda^{-1} \circ \tilde{r})(0) = 0$, $(\lambda^{-1} \circ \tilde{r})(\infty) = \infty$, and there are no other points where $\lambda^{-1}(\tilde{r})$ takes values 0 and $\infty$, so $r = \lambda^{-1}(\tilde{r})$ is an algebraic polynomial of the form $z^k$ for some natural $k$.

2) If $\tilde{P}$ has only one preimage of $\infty$, then similarly to the lemma 2.9 there exists a linear rational function $\lambda$ s.t. $\bar{P} = \tilde{P}(\lambda)$ is a polynomial. Then necessary $(\lambda^{-1} \circ \tilde{r})(0) = \infty$, $\quad (\lambda^{-1} \circ \tilde{r})(\infty) = \infty$, and there are no other points where $\lambda^{-1}(\tilde{r})$ takes values 0 and $\infty$, so $\lambda^{-1} \circ \tilde{r}$ is a Laurent polynomial.

Since under composition with a linear function the number of preimages of a given point can not change, (1) and (2) are not equivalent. ■

**Corollary 7.4.1** *If* $\deg(P, Q) > 1$, *then* $P = \bar{P}(r)$, $Q = \bar{Q}(r)$ *with either*
*(1)* **Laurent polynomial composition**: $\bar{P}$, $\bar{Q}$ *– algebraic polynomials,* $r$ *– Laurent polynomial of degree greater than 1; or*
*(2)* $\bar{P}$, $\bar{Q}$ *– Laurent polynomials,* $r = z^k$ *for* $k \geq 2$.

# Chapter 8

# Moments of $P$, $Q$ on $S^1$ and Center Conditions for Abel equation with rational $p$, $q$

## 8.1 Introduction

The role of generalized moments of the form $\int P^i Q^j dP$ and $\int P^i q\, dz$, as related to the center-Focus problem and composition conditions on the interval, is investigated in [BFY2-4], [BY2]. Recently a serious progress have been done in this direction ([Pa1-2], [N], [Chr2]).

In this thesis we restrict ourselves to the case of Laurent polynomials, and generalized moments on the circle.

It is shown that certain integral condition on $P = \int p$ and $Q = \int q$ (namely vanishing of **generalized moments** $\int_{|z|=1} P^i Q^j dP = 0$) implies center.

## 8.2 Sufficient center condition for Abel equation with analytic $p$, $q$

We return to the case of Abel equation (A)

$$dy = y^2 dP + y^3 dQ \qquad (A)$$

considered in a neighborhood of a unit circle $S^1 = \{|x| = 1\}$ in the complex plane $\mathbb{C}$, with $P$, $Q$ – analytic functions in some neighborhood of $S^1$ (not necessary Laurent polynomials).

The following theorem is a summary of results due to J. Wermer ([W1], [W2]). The applicability of Wermer's results to Center problem was discovered by J.-P.Françoise ([F]):

**Theorem (Wermer, 1958)** *Let $P$, $Q$ be a pair of functions on the unit circle $S^1 \subseteq \mathbb{C}$. Assume:*
*(1) $P$ and $Q$ are analytic in an annulus containing $S^1$ and together separate points on $S^1$.*
*(2) $P' \neq 0$ on $S^1$.*
*(3) $P$ takes only finitely many values more than once on $S^1$.*

*If $\int_{s^1} P^i Q^j dP = 0$ for all $i, j \geq 0$, then there exists a Riemann Surface $X$ and a homeomorphism $\varphi : S^1 \to X$, such that $\varphi(S^1)$ is a simple closed curve on $X$ bounding a compact region $D$, such that functions $\tilde{P}$, $\tilde{Q}$ defined on $\varphi(S^1)$ by $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ can be extended inside $D$ to be analytic there and continuous in $D \cup \varphi(S^1)$.*

To use this theorem for our factorization, we need to replace "homeomorphism" by "analytic map of a certain neighborhood of $S^1$ into $X$".

**Lemma 8.2.1** *Let $S^1 \subseteq \mathbb{C}$ be a unit circle, $P$ and $Q$ be analytic functions in a neighborhood $U$ of $S^1$, such that $P' \neq 0$ on $S^1$. Let $X$ be a Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ – regular functions on $X$, and let $\varphi : S^1 \to X$ be a homeomorphism such that $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ on $S^1$. Then $\varphi$ can be extended as an analytic mapping of a certain neighborhood $V \subseteq U$ of $S^1$ into $X$, with the same property $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ in $V$.*

**Proof:**
$P'(s) \neq 0$, hence $\tilde{P}'(\varphi(s)) \neq 0$, so $\tilde{P}'(y) \neq 0$ in a neighborhood of $\varphi(s)$ on $X$. We define $\varphi(x)$ in this neighborhood as $y = \varphi(x) = \tilde{P}^{-1}(P(x))$. Locally $\varphi$ exists and is well-defined. Since these local extensions agree on $S^1$, they in fact agree and define a required extension on a certain neighborhood of $S^1$. ■

**Corollary 8.2.1** *If in the Abel equation (A) on $\mathbb{C}$*

$$dy = y^2 dP + y^3 dQ \quad or \quad dy = y^2 dQ + y^3 dP$$

*P and Q are functions, satisfying all the properties* (1) – (4) *and the domain D, provided by Wermer's theorem, is simply-connected, then the Abel equation (A) has a center.*

**Proof:**
If $P' \neq 0$ on $S^1$, we may apply lemma 4.1. Then the Abel equation (A) is induced by the analytic mapping $\varphi$ from the Abel Equation on $X$

$$dy = y^2 d\tilde{P} + y^3 d\tilde{Q}. \qquad (\tilde{A})$$

Since $\tilde{P}$, $\tilde{Q}$ are analytic on a simply connected domain $D$ bounded by $\varphi(S^1)$, the equation $(\tilde{A})$ has a center along $\varphi(S^1)$, and hence the equation (A) has a center along $S^1$. ■

Notice, that this condition is a sufficient condition for center for Abel equation with arbitrary analytic coefficients. But it is symmetric with respect to $P$ and $Q$, although some of the center conditions for Abel equation are known to be non-symmetric. Below we shall explain it for the case of $P$, $Q$ – Laurent polynomials.

## 8.3 The degree of a rational mapping and an image of a circle on a rational curve

Consider two rational functions $P$, $Q$. The map $\gamma = [P, Q] : \mathbb{C} \to \mathbb{C}^2$ defines the rational curve $Y = \{(P(t), Q(t)) : t \in \mathbb{C}\}$. Image of a circle $S^1$ under the map $\gamma$ is a closed curve on $Y$.

**Theorem 8.3.1** *Let $P$, $Q$ be rational functions without poles on $S^1$, s.t. at least one of them has a pole inside $S^1$ and at least one of them has a pole outside $S^1$ (for instance, Laurent polynomials). Let $\gamma(S^1)$ bound a compact domain in $Y$. Then* $\deg \gamma > 1$.

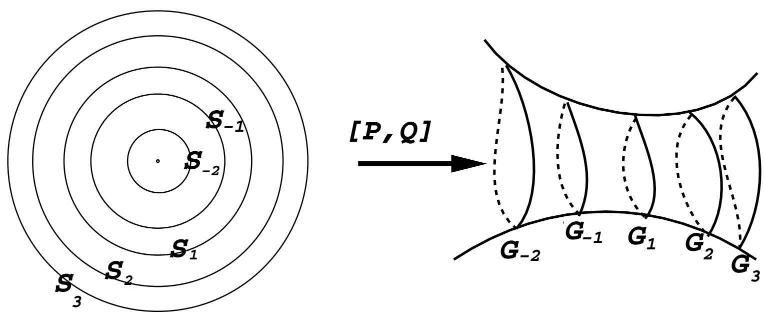
**Proof:**
Assume that $\deg \gamma = 1$. Then consider a path $\chi$ in $\mathbb{C}$, joining two poles of $P$ and $Q$ inside and outside of $S^1$ ( for simplicity 0 and $\infty$), and intersecting $S^1$ only once at a *regular* point $u \in \gamma$. We can assume also that $\chi$ does

not contain preimages of double points in $Y$. So for any $x \in \chi$ there are no $y \neq x \in \mathbb{C}$ with $\gamma(x) = \gamma(y)$.

$\gamma(\chi(z))$ tends to $\infty$ as $z$ tends to 0 and to $\infty$, so the image of $\chi(z)$ under the map $\gamma : \mathbb{C} \to Y$ can not stay inside a compact domain bounded by $\gamma(S^1)$. But it enters this domain, since $u$ is a regular point of $\gamma$. Then it must intersect $\gamma(S^1)$ at another point $v \neq u$, and we get contradiction to the choice of the path $\gamma$. ■

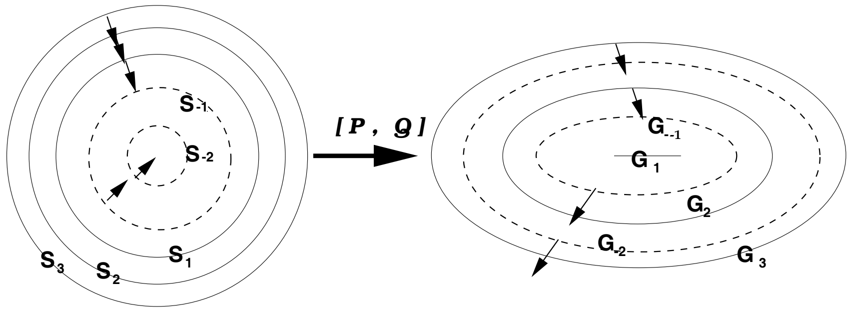
**Example 1.** $P(z) = z, \quad Q(z) = \dfrac{1}{z}.$

The rational curve $Y$ is $\{xy = 1\} \subseteq \mathbb{C}^2$, and the curve $\gamma(S^1)$ does not bound a compact domain on it. The degree of the map $[P, Q]$ is one, and it is a general situation for a map of degree one: images of circles contracted to poles diverge on $Y$ – see figure 1 with $S_k = \{|z| = k\}$, $S_{-k} = \{|z| = \frac{1}{k}\}$ ($k \in \mathbb{N}$), and $G_k$, $G_{-k}$ – their images on $Y$ under $\gamma = [P, Q]$.




**Example 2.** $P(z) = z + \dfrac{1}{z}, \quad Q(z) = z + \dfrac{1}{z}.$

The rational curve $Y$ is $\{x = y\} \simeq \mathbb{C}$, the degree of the map $\gamma$ is 2, and the curve $\gamma(S^1)$ bounds a compact domain on $Y$ (in fact, $S_1 = \gamma(S^1) = [-1, 1]$). See figure 2 for illustration: images of the circles $S_k$ cover $Y$ twice, because they "have no space to diverge". Arrows indicate directions of "motion" of the curves $S_k$ and $G_k$ as $k$ decreases from $+\infty$ to $-\infty$.




## 8.4 Center conditions for the case of $P$, $Q$ – Laurent polynomials

**Theorem 8.4.1** *If $P$ and $Q$ are Laurent polynomials, satisfying the condition*

$$\int_{|z|=1} P(z)^k Q(z)^n dP = 0$$

*for all pairs (k,n) of nonnegative integers, and $P$, $Q$ together separate points on $S^1$, then $P$ and $Q$ can be represented in the form of* **Laurent Polynomial composition**, *and hence the equation (A) has a center.*

**Remark:** We believe that for the case of Laurent polynomials the Wermer's theorem remains valid without the assumption that $P$ and $Q$ together separate points on $S^1$. We hope to present a proof of this fact together with a detailed investigation of Wermer's surface for the case of $P$, $Q$ – Laurent polynomials.

**Proof:**
If both $P$ and $Q$ are algebraic polynomials in $z$, $P$ and $Q$ are represented as a composition with $z$, so we have a center.

Similarly, if both $P$ and $Q$ are algebraic polynomials in $\frac{1}{z}$, $P$ and $Q$ are represented as a composition with $\frac{1}{z}$, so we have a center. Otherwise $P$ and $Q$ have one pole inside (origin $z = 0$) and one pole outside (infinity) of $S^1$.

Obviously $P$, $Q$ are analytic in a neighborhood of $S^1$ and take only finitely many values more than once on $S^1$.

Next, we always can assume that $P'(z) \neq 0$ for all $|z| = 1$. Indeed, (A) after the change of variables $z = \lambda u$ became

$$\frac{dy}{du} = d\hat{P}(u)y^2 + d\hat{Q}(u)y^3, \qquad (\hat{A})$$

where $\hat{P}(u) = \lambda P(\lambda u)$, $\hat{Q}(u) = \lambda Q(\lambda u)$, both $(A)$ and $(\hat{A})$ have center simultaneously. But as $P'$ has only finite number of zeroes on $\mathbb{C}$, by rescaling $z \mapsto \lambda z$, which does not change a center for (A), we can assure that there are no zeroes of $\hat{P}'$ on the circle $S^1$. For example, $P(z) = z + \frac{1}{z}$ has zeroes of $P'$ on $S^1$, but $\hat{P}(z) = 4z - \frac{1}{z}$ has not. Under this change of variables the circle $|z| = 1$ goes to $|u| = \lambda$, but by Cauchy theorem for Laurent polynomials integrals along these circles coincide.

Hence by Wermer's theorem there exists a surface $X$ and a homeomorphism $\varphi : S^1 \to X$, such that $\varphi(S^1)$ bounds a compact domain $D$ on $X$ and there exists functions $\tilde{P}$, $\tilde{Q}$ analytic inside $D$..

Remind that we have a map $\gamma = [P, Q] : \mathbb{C} \to Y = \{(P(t), Q(t)) : t \in \mathbb{C}\}$.

**Lemma 8.4.1** *If the curve $\varphi(S^1)$ bounds a compact domain on $X$, then the curve $\gamma(S^1)$ bounds a compact domain on a rational curve $Y$.*

**Proof:**
By lemma 4.1 $[P, Q] \circ \varphi$ is an analytic mapping, defined in a neighborhood of $S^1$, which coincides there with $\gamma = [P, Q]$. But hence $[\tilde{P}, \tilde{Q}] : X \to \mathbb{C}^2$ maps a neighborhood of $\varphi(S^1)$ into $Y \subseteq \mathbb{C}^2$. By analytic continuation, $[\tilde{P}, \tilde{Q}]$ maps $X$ into $Y$. Hence $[\tilde{P}, \tilde{Q}]$ maps a compact domain $D$ inside $\varphi(S^1)$ onto $Y$, and the image of a compact domain under continuous mapping is compact. Hence $\gamma(S^1)$ is contained in a compact $[\tilde{P}, \tilde{Q}](D)$. Now one can easily show that in fact $\gamma(S^1)$ bounds a compact domain in $Y$. ■

**Proof of theorem 4.2 (continue):**
By theorem 4.1 $\deg[P, Q] > 1$, hence $P$ and $Q$ can be represented as a composition $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$. If $\tilde{P}$ and $\tilde{Q}$ are usual algebraic polynomials, we are done.

If not, then $P(z) = \tilde{P}(z^k)$, $Q(z) = \tilde{Q}(z^k)$ for Laurent polynomials $\tilde{P}$, $\tilde{Q}$. But then on the rational curve $Y$ we get $[P, Q](S^1) = [\tilde{P}, \tilde{Q}](S^1)$, so $[\tilde{P}, \tilde{Q}](S^1)$ bounds a compact domain on $Y = \{(P(t), Q(t)) : t \in \mathbb{C}\} =$

$\{(\tilde{P}(t), \tilde{Q}(t)) : t \in \mathbb{C}\}$. Therefore by theorem 4.1 $\deg[\tilde{P}, \tilde{Q}] > 1$, so they are represented as a composition.

If we again obtain their representation as a composition of Laurent polynomials with $z^n$, we repeat our considerations, and finally we are left with the composition $\tilde{P} = \tilde{\tilde{P}}(\tilde{w})$, $\tilde{Q} = \tilde{\tilde{Q}}(\tilde{w})$ with $\tilde{w}$ – Laurent polynomial, $\tilde{\tilde{P}}$ and $\tilde{\tilde{Q}}$ – algebraic polynomials. It gives us the composition $P = \tilde{\tilde{P}}(\tilde{w}(z^N))$, $Q = \tilde{\tilde{Q}}(\tilde{w}(z^N))$, and we are done. ■

# Chapter 9

# Generalized Moments and Center Conditions for Abel differential equation with elliptic $p$, $q$

## 9.1 Introduction

In Chapter 8 we considered Generalized moments for rational functions $P$ and $Q$ and have shown that vanishing of all generalized moments implies composition representability of $P$ and $Q$ and hence center for Abel equation (A):

$$dy = y^2 dP + y^3 dQ. \qquad (A)$$

In the proof we essentially used the fact that $[P,Q](\mathbb{C})$ is a rational curve, i.e. algebraic curve of genus 0.

Suggested generalization is to consider the same question for Laurent series instead of Laurent polynomials. The first nontrivial case of Laurent series in our "algebraic" context is the case of elliptic functions. For $P$, $Q$ – elliptic functions we show that all the generalized moments $\int P^i Q^j dP$ along a small curve around the origin vanish. But at the same time we demonstrate that in general Abel equation (A) with $P$, $Q$ – elliptic functions does not have a center, in spite of the fact that all the generalized moments on $S^1$ vanish. So an analog of the theorem from [BY2] for infinite Laurent series does not hold.

It can be explained by the fact that elliptic curve $X = [P, Q](\mathbb{C})$ is an algebraic curve of genus 1 and hence it is homologically nontrivial. $S^1$ is embedded by the map $(P, Q)$ into this curve in a homologically trivial way, and hence all the moments on $S^1$ vanish. But the domain, bounded by $(P, Q)(S^1)$ is not simply-connected. Hence multiple integrals along this curve do not vanish, and (A) does not have center.

We start from the simplest case when $P$ is the Weierstrass $\rho$-function, $Q$ is it's derivative. In this case the surface $[P, Q](\mathbb{C})$ is a torus with small circle around the origin dividing it into two parts: compact *topologically nontrivial* part, on which $\rho$ and $\rho'$ are regular (have no poles), and the second part wich is topologically equivalent to $\mathbb{C}$ without origin, but $\rho$ and $\rho'$ have a pole at zero.

picture

## 9.2 Generalized Moments on Elliptic Curves

As it was shown above (Chapter 3), the study of the center problem for the Abel equation leads naturally to the "moment-like" expressions of the form $\int P^i\, dQ$ and $\int Q^i\, dP$, and similar more complicated ones.

All these expressions naturally appear as a part of the expressions for the Taylor coefficients of the Poincare mapping given in Chapter 3 above.

Transforming these coefficients via integration by parts, one obtains expressions of the form $\int P^i\, Q^j\, dP$ and $\int P^i\, Q^j\, dQ$, and the expressions

$$m_{i_1 i_2 \ldots i_n}(x) = \int_0^x P^{i_1}(x_1) q(x_1) \int_0^{x_2} P^{i_2}(x_2) q(x_2) \ldots \int_0^{x_{n-1}} P^{i_n}(x_n) q(x_n)\, dx_n \ldots dx_1,$$

where $p(x) = dP(x)/dx$, $q(x) = dQ(x)/dx$ (introduced by J-P. Francoise).

Expressions of this form, considered on complex curves of genus greater than zero, lead to multivalued functions. Hence we generalize definition of moments to this case:

**Definition 9.2.1** *For the two multivalued functions $P$, $Q$ on a Riemann Surface $X$ we define* **generalized moments** $m_{i,j}$ *and* $m'_{i,j}$ *as integrals along a given curve* $\gamma$ *of*

$$dm_{i,j} = P^i Q^j dQ \ \ (i, j \geq 0) \tag{9.1}$$

$$dm'_{i,j} = P^i Q^j dP \ \ (i, j \geq 0) \tag{9.2}$$

*We normalize* $m_{i,j}$ *and* $m'_{i,j}$, *assuming that all of them vanish at a fixed point* $a$. *Here* $P$ *and* $Q$ *are analytically continued along the curve* $\gamma$ *from the point* $a$.

**Remark 9.2.1:** Let us notice, that

$$dm'_{i,j} = Q^j d\left(\frac{P^{i+1}}{i+1}\right) = d\left(\frac{Q^j P^{i+1}}{i+1}\right) - \frac{P^{i+1}}{i+1} dQ^j = d\left(\frac{Q^j P^{i+1}}{i+1}\right) - \frac{j}{i+1} dm_{i+1,j-1}$$

Below we shall use this relation in computations of $m_{i,j}$.

Following [BFY3-4], we can consider a problem of vanishing (or non-ramification) of generalized moments:

**Moment center problem:** *Find conditions on* $P$, $Q$ *under which all the moments* $m_{k,n}(x)$ *for a certain set of indices* $(i, j)$ *are univalued functions along a given curve* $\gamma$ *(and if* $\gamma$ *is not closed with end points* $a$ *and* $b$, *then* $a$ *and* $b$ *are conjugate, i.e.* $m_{i,j}(a) = m_{i,j}(b)$ *under analytic continuation along* $\gamma$*).*

Below we present some rather preliminary computations in this direction for elliptic functions.

**Theorem 9.2.1** *Let* $P$, $Q$ *be a pair of elliptic functions with the only pole at zero. Then for all nonnegative* $k$, $n$ *and all sufficiently small* $\alpha$ *(inside the parallelogramm of periods) the generalized moments*

$$\int_{|u|=\alpha} P(u)^k Q(u)^n dP \tag{9.3}$$

*vanish.*

**Proof:** The proof follows from the fact, that for sufficiently small $\alpha$ (smaller than $\min(\omega_1, \omega_2)$) elliptic functions $P$ and $Q$ do not have poles (in the parallelogramm of periods) outside the circle $|u| = \alpha$. Hence, the circle $|u| = \alpha$ bounds (on the elliptic curve) a compact domain, on which the integrand is regular. ∎

**Remark:** We may show this fact, using combinatorial arguments. Remind some basic notions from the theory of Weierstrass $\rho$-function. In local coordinates $z$ around zero on $\mathbb{C}$ it can be defined by the formula

$$\rho(z) = \frac{1}{z^2} + \sum_{i=2}^{\infty} c_i \; z^{2i-2},$$

where coefficients $c_i$ are computed by recursion

$$c_i = 3\frac{\sum_{j=2}^{i-2} c_j \; c_{i-j}}{(i-3)(2i+1)}.$$

Here the first coefficients are defined as $c_2 = g_2/20$, $c_3 = g_3/28$, where $g_2$ and $g_3$ depend only on periods $\omega$:

$$g_2 = 60 \sum \frac{1}{\omega^4}, \quad g_3 = 140 \sum \frac{1}{\omega^6},$$

where summation is taken over all non-zero peiods of $\rho$.

The following combinatorial formulas hold:

$$(\rho')^2 = 4\rho^3 - g_2\rho - g_3, \quad \rho'' = 6\rho^2 - \frac{g_2}{2}.$$

Now the proof of the theorem 9.2.1 follows from the fact that any integral of the form (9.3) can be presented as a sum of integrals of the form $\int_{|u|=\alpha} \rho(u)^k \rho(u)' du$ and $\int_{|u|=\alpha} \rho^k(u) du$. Both integrals are zeros: the first being the increment of $\frac{1}{k+1}\rho'$ along $|u| = \alpha$, while the second contains only even degrees of $z$ and hence does not have residue inside $|u| = \alpha$.

Let us notice, that from remark 9.2.1 follows that generalized moments $\int_{|u|=\alpha} P(u)^k Q(u)^n dP$ and $\int_{|u|=\alpha} P(u)^k Q(u)^n dQ$ are in equal, as $\int d\left(\frac{Q^j P^{i+1}}{i+1}\right) =$

0.

Next question we are interested in is computation of generalized moments along periods of elliptic integrals.

**Theorem 9.2.2** *Let $\rho$ be Weierstrass $\rho$-function, $\omega_1$, $\omega_2$ be periods. Then*

$$\int_{\omega_i} \rho(u)^n \rho'(u)^{2k+1} du = 0.$$

*for all $k$ and $n$.*

**Proof:** $\int_{\omega_i} \rho^n \rho'^{2k+1} du = \int_{\omega_i} \rho^n (4\rho^3 - g_2\rho - g_3)^k \rho' du = \int_{\omega_i} F(\rho)\rho' du = 0.$ ■

WILL BE ADDED:

Computations for $\int_{\omega_i} \rho(u)^n \rho'(u)^{2k}$. It is known that in general generalized moments along period is equal to:

$$\int_{\omega_i} = A\omega_i + B\eta_i,$$

where $\eta_i = \int_{\omega_i} \rho(u) du$. we'll find these $A$ and $B$.

## 9.3 Abel equation with coefficients–elliptic functions

As we saw in the previous section, all the generalized moments $\int_{|u|=\alpha} P(u)^k Q(u)^n dP$ for elliptic functions $P$, $Q$ with the only pole at the origin vanish. For rational functions that would imply center (under some additional restrictions on $P$ and $Q$). For elliptic functions the similar statement does not hold. Although for some elliptic $P$, $Q$ the equation (A) may have center (say, $P = Q$), in general (A) does not have a center. In particular, the following theorem holds:

**Theorem 9.3.1** *Let $\rho$ be Weierstrass $\rho$-function. The equations*

$$dy = y^2 d\rho + y^3 d\rho' \text{ and} \tag{9.4}$$

$$dy = y^2 d\rho' + y^3 d\rho \tag{9.5}$$

*ramify along any small circle around zero.*

**Proof:** By theorem 6.3.2. (Chapter 6) in order for Abel differential equation (A) to have center all the functions $\varphi_k(z)$ in the expansion of Poincare Return Map must be univalued functions of $z$. Substituting finite expansion (for sufficiently many $c_i$ to garantee all the necessary terms for all negative powers of $z$) for $\rho(z)$ into formulas of Chapter 3 for $\varphi_k(z)$, we find that for Abel equation (9.4) $\varphi_8(z)$ has logarithmic term $\frac{2}{105}(g_2^3 - 27g_3^2)\ln z$, which is the determinant of the Weierstrass function $\rho$ and hence nonzero, i.e. $\varphi_8(z)$ is non-univalued function.

For Abel equation (9.5) all terms up to $\varphi_8(z)$ are univalued functions, but $\varphi_9(z)$ has logarithmic term $\frac{2}{1155}(g_2^3 - 27g_3^2)(6g_2 - 11)\ln z$, which can be zero only for $g_2 = 6/11$. However, if $g_2 = 6/11$, $\varphi_{11}(z)$ has logarithmic term $\frac{12}{85085}g_3(g_2^3 - 27g_3^2)\ln z$, and it is nonzero. ■

WILL BE ADDED:

**Remark 1:** We'll add handwriting proof of this fact. Also: explanation, why (9.4) and (9.5) have different $\varphi_k$ vanishing (because of $\rho'$).

**Remark 2:** Another remark is that in contrast with the notions of center and conjugate points for the Abel equation on a plane, where all the definitions depend on *homotopy class* of the path $\gamma$, here we meet with *homology class* of $\gamma$. Here we integrate the closed forms $P^iQ^j\,dP$, $P^iQ^j\,dQ$ several times.

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