Some computations around the center problem , related to the algebra of univariate polynomials

Thesis for the M.Sc. Degree

by

Michael Blinov

Under the Supervision of professor
Yosef Yomdin
Department of Theoretical Mathematics
The Weizmann Institute of Science

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the Weizmann Institute of Science
Rehovot 76100, Israel

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1 Research objectives

1.1 Introduction

In [7] H.Poincaré defined the notion of a center for a real vector field on the plane

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

as an isolated singularity surrounded by closed integral curves. He showed (see [8]) that a necessary and sufficient condition for a polynomial vector field (i.e. $f(x,y) = P(x,y)$, $g(x,y) = Q(x,y)$ are polynomials in x, y) with a singular point with pure imaginary eigenvalues, to have a center at this point is the annihilation of an infinite number of polynomials in the coefficient of the vector field. The problem of explicitly finding a finite basis for these algebraic conditions (the problem of the center), was solved in the case of quadratic vector fields by the successive contributions of H.Dulac, W.Kapteyn, N.Bautin, N.Sakharnikov, L.Belyustina, K.Sibirsky and others (see e.g. [1], [9]). The complete conditions on $P(x,y)$, $Q(x,y)$ of degrees higher then 2 under which the system has a center are still unknown.

1.2 Description of the problem

1.2.1 The center problem

We will consider the following formulation of the center problem: Let $P(x,y)$, $Q(x,y)$ be polynomials in x, y of degree d. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + P(x,y) \\ \dot{y} = x + Q(x,y) \end{cases} \tag{1}$$

We will say that a solution $x(t)$, $y(t)$ of (1) is closed if it is defined in the interval $[0, t_0]$ and $x(0) = x(t_0)$, $y(0) = y(t_0)$. We will say that the system (1) has a center at 0 if all the solutions around zero are closed. Then the general problem is: under what conditions on P, Q the system (1) has a center at zero?

1.2.2 Reduction to the Abel equation

It was shown in [4] that one can reduce the system (1) with homogeneous P, Q of degree d to the Abel equation

$$y' = p(x)y^2 + q(x)y^3 \tag{2}$$

where $p(x)$, $q(x)$ are polynomials in $\sin x$, $\cos x$ of degree depending only on d. Then (1) has a center if and only if (2) has periodic solutions on $[0, 2\pi]$, i.e. solutions $y = y(x)$ satisfying $y(0) = y(2\pi)$.

1.2.3 Classical approach to the study of the Abel equation

We will look for solutions of (2) in the form

$$y(x, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(x) y_0^k,$$

where $y(0, y_0) = y_0$. Then $y(2\pi) = y(2\pi, y_0) = y_0 + \sum_{k=2}^{\infty} v_k(2\pi) y_0^k$. Then the condition $y(2\pi) \equiv y(0)$ is equivalent to $v_k(2\pi) = 0$ for $k = 2, 3, \dots \infty$.

Consider an ideal $J = \{v_2(2\pi), v_3(2\pi), \dots v_k(2\pi), \dots\} \subseteq C[\lambda]$, where $\lambda = (\lambda_1, \lambda_2, \dots)$ is the (finite) set of the coefficients of p, q. By Hilbert Basis theorem there exists $d_0 < \infty$, s.t. $J = \{v_2(2\pi), v_3(2\pi), \dots v_{d_0}(2\pi)\}$. After determination of d_0 the general problem will be solved, since we get finite number of conditions on λ, which define the set of p, q having all the solutions closed . The problem is that the Hilbert theorem does not allow us to define d_0 constructively.

1.2.4 Modified approach to the study of the Abel equation

Let us study instead of $J \subseteq C[\lambda]$ the polynomials ideal $I \subseteq C[\lambda, x]$, $I = \{v_2(x), v_3(x), \dots v_k(x), \dots\} = \bigcup_{k=2}^{\infty} I_k$, where $I_k = \{v_2(x), v_3(x), \dots v_k(x)\}$. The classical problem is to find conditions on p, q, under which $x = 2\pi$ is a common zero of all I_k.

Our **generalized center problem** consists of the following:

a) Study the behavior of I_k as the ideals of univariate polynomials in x, i.e.

i. For given p, q find zeroes in x of I_k, $k = 2, \dots$ and of $I = \bigcup_{k=2}^{\infty} I_k$.

ii. For a given set of numbers find conditions on p, q, under which these numbers are common zeroes of I.

b) Find the stabilization moment of the set of common zeroes, i.e.

i. For given p, q find d, for which the set of common zeroes of I is equal to the set of common zeroes of I_d.

ii. For given set of common zeroes of I find d, for which it is equal to the set of common zeroes of I_d. Under which conditions on p, q is it possible?

c) For given p, q find d, for which $I = I_d$ (Bautin's problem).

1.2.5 Main recurrence relations

We study Abel equation (2) with p, q the usual polynomials in x. In this case we say that the equation (2) defines a center if $y(1, y_0) \equiv y_0$. Although this property does not correspond to the initial problem (1), it presents an interest by itself and it has been studied in [5], [6] and in many others articles. Our main goal is to study the generalized center problem for this case, our first goal is to study part a) of it.

One can easily show (see e.g. [2]) that $v_k(x)$ satisfy recurrence relations

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ v_n'(x) = p(x) \sum\limits_{i+j=n} v_i(x)v_j(x) + q(x) \sum\limits_{i+j+k=n} v_i(x)v_j(x)v_k(x), \;\; n \geq 2 \end{cases} \tag{3}$$

It was shown in [2] that in fact this recurrence relations can be linearized, i.e. the same ideals I_k's are generated by $\{\xi_1, \dots \xi_k\}$, where $\xi_k(x)$ satisfy linear recurrence relations

$$\boxed{\begin{cases} \psi_0(x) \equiv 0 \\ \\ \psi_1(x) \equiv 1 \\ \\ \psi_n'(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \ n \geq 2 \end{cases}} \tag{4}$$

which are much more convenient then (3). We call (4) **the main recurrence relation** for the main problem.

1.2.6 Model problem

Let us state an auxiliary problem:

The first model problem : Given $p(x)$, $Q_0(x)$. Define $Q_{k+1}(x)$ by recurrence relation $Q'_{k+1}(x) = p(x)Q_k(x)$, $k \geq 0$. Study the generalized center problem for the ideals $I_k = \{Q_0(x), \ldots, Q_k(x)\}$.

Hopefully this problem can help us to study the main problem (4). It allows for "analytic" solutions (through generating functions).

For the main problem the first few ideals I_k are very similar to the first few ideals of the first model problem , but starting with the I_6 essentially nonlinear equations with respect to $Q(x) = \displaystyle\int_0^x q(t)dt$ appear. This fact presents the main difficulty in analysis of the problem (4).

1.2.7 Main conjecture and known results

The following conjecture for the main problem (4) has been proposed in [2]:

$$I = \bigcup_{k=1}^{\infty} I_k \text{ has zeroes } a_1, \ldots a_k \text{ if and only if}$$
$$P(x) = \int_0^x p(t)dt = \tilde{P}(W(x)) \ , \ Q(x) = \int_0^x q(t)dt = \tilde{Q}(W(x)) \ ,$$
where $W(x) = \prod_{i=1}^{k}(x - a_i)$,
$\tilde{P}$, $\tilde{Q}$ are some polynomials without free terms.

Exactly the same conjecture can be stated for the first model problem, with $Q = Q_0$. Clearly, these conjectures are sufficient for zeroes of W to be common zeroes of I in each of these problems. It is not clear yet if these conditions are also necessary.

The following particular results are known:

1) The conjecture is true for $P(x)$, $Q(x)$ up to degree 3 and for some cases of degree 4 (see [2]).

2) For the first model problem if $P(x) = W(x) = \prod_{i=1}^{k}(x - a_i)$, then $\bigcup_{k=1}^{\infty} I_k$ has zeroes $a_1, \ldots a_k$ if and only if $Q_0(x) = \tilde{Q}_0(W(x))$ (see [3]).

3) For the first model problem combinatorial estimation of the I_k's stabilization moment is obtained (see [10], [11]).

1.2.8 Results

We present the following results:

a) Some remarks, connecting to the first model problem. They can be useful as a tool for an estimation of the number of surviving zeroes (section2).

b) We have obtained the number of conditions, which should be checked in order to say, that the hypothesys for the first model problem is true, and some remarks about sufficiency of this number (section 3).

c) Maximal number of zeroes of I for the recurrency relation (4) is obtained (section 4).

d) Verification of the main conjecture 2.7 for the main problem (4) with the degrees of P , Q up to 4 and higher . It was done using computer symbolic calculations with some convenient representation of P and Q (section 5).

2 Some remarks around the first model problem.

Consider so called **"the zero model problem" :** Given $\phi_0(x)$. Define $\phi_{k+1}(x)$ by recurrence relation $\phi'_{k+1}(x) = \phi_k(x)$,$\phi_{k+1}(0) = 0$, $k \geq 0$. Study the generalized center problem for the ideals $I_k = \{\phi_0(x), \ldots, \phi_k(x)\}$.

3 ???

3.1 A convenient representation of P and Q.

Assume we are interested in the checking if numbers 0, a are common zeroes of our ideal I. Let $R(x)$ be an arbitrary polynomial of degree n. Consider $W(x) = x(x-a)$ - polynomial of the second degree. Notice, that the derivative of W is a polynomial of the first degree $W'(x)$, the polynomial $W(x)W'(x)$ has the second degree and so on. Generally, polynomials $W(x)^k$ have degree $2k$ and polynomials $W(x)^k\ W'(x)$ have degree $2k+1$. So, one can uniquely represent any polynomial $R(x)$ as a linear combination of polynomials $W(x)^k$ and $W(x)^k W'(x)$. Hence the polynomial $R(x)$ of the degree $2k$ or $2k+1$ we will write in the form

$$R(x) = W(x)^k\,(\alpha_k W(x)' + \beta_k) + W(x)^{k-1}\,(\alpha_{k-1} W(x)' + \beta_{k-1}) + ... + (\alpha_0 W(x)' + \beta_0),$$

or simply

$$R(x) = W^k(\alpha_k W' + \beta_k) + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0).$$

In general, if $W(x) = x(x-a_2)\dots(x-a_k), \deg W(x) = k$, then any polynomial $R(x)$ can be uniquely represented in the form

$$R(x) = W^m(c_m^1 W' + c_m^2 W'' + \dots + c_m^k W^{(k)}) + \dots + (c_0^1 W' + c_0^2 W'' + \dots + c_0^k W^{(k)}),$$

(where, of course, $W^{(k)}$ is simply constant).

4 Maximal number of surviving zeroes.

4.1 Connection between the first model problem and the main problem. A convenient basis for the ideals $I_k, k = 2, \dots 6$.

Direct computations (including several integrations by part) give the following expressions for the first polynomials $\psi_k(x)$, solving the recurrency relation (4):

$$\begin{aligned}
\psi_2(x) &= -P(x) \\
\psi_3(x) &= P^2(x) - Q(x) \\
\psi_4(x) &= -P^3(x) + 3P(x)Q(x) - \int_0^x q(t)P(t)dt \\
\psi_5(x) &= P^4(x) - 6P^2(x)Q(x) - \int_0^x q(t)P^2(t)dt \\
&\quad +4P(x)\int_0^x q(t)P(t)dt + \frac{3}{2}Q^2(x) \\
\psi_6(x) &= -P^5(x) + 10P^3(x)Q(x) + 5P(x)\int_0^x q(t)P^2(t)dt \\
&\quad -8Q^2(x)P(x) - 10P^2(x)\int_0^x q(t)P(t)dt + 4Q(x)\int_0^x q(t)P(t)dt \\
&\quad -\int_0^x q(t)P^3(t)dt + \frac{1}{2}\int_0^x p(t)Q^2(t)dt
\end{aligned}$$

Consequently, we get the following set of generators for the ideals $\tilde{I}_k$, $k = 2, \dots, 6$,

$$\begin{aligned}
I_2 &= \{P\} \\
I_3 &= \{P, Q\} \\
I_4 &= \{P, Q, \int qP\} \\
I_5 &= \{P, Q, \int qP, \int qP^2\} \\
I_6 &= \{P, Q, \int qP, \int qP^2, \int (qP^3 - \frac{1}{2}pQ^2)\}
\end{aligned}$$

Therefore, if $a \in Y(\tilde{I}_6)$ is a zero of the ideal $\tilde{I}_6$, it must satisfy the following equations:

$$P(a) = 0\ ,\ \ Q(a) = 0$$

$$\int_0^a P(t)q(t)dt = 0 \tag{5}$$

$$\int_0^a P^2(t)q(t)dt = 0$$

$$\int_0^a P^3(t)q(t)dt - \frac{1}{2}\int_0^a p(t)Q^3(t)dt = 0$$

Notice that the third and the fourth equations coincide with the moment equations of the first model problem (with the same $p(x)$ and $Q_0(x) = Q(x)$). The fifth equation contains the corresponding term of the model problem and an additional term, which is (for the first time) nonlinear in Q.

Let us assume now that the set of zeroes of $\tilde{I}_6$ consists of the points $a_1 = 0, a_2, \ldots, a_\nu$, $a_i \neq a_j$. In particular, a_i are common zeroes of P and Q, and we can write

$$P(x) = W(x)P_1(x)Q(x) = W(x)Q_1(x)$$

where $W(x) = \prod_{i=1}^{\nu}(x - a_i)$.

Substituting this representation into the last three equations of (5) and integrating by parts, we get for $i = 1, \ldots, \nu$,

$$\int_0^{a_i} W^2(p_1Q_1 - P_1q_1) = 0$$

$$\int_0^{a_i} W^3P_1(p_1Q_1 - P_1q_1) = 0$$

$$\int_0^{a_i} W^4P_1^2(p_1Q_1 - P_1q_1) - \frac{2}{3}\int_0^{a_i} W^3Q_1(p_1Q_1 - P_1q_1) = 0$$

Here $p_1(x) = P_1'(x)$, $q_1(x) = Q_1'(x)$.

4.2 Maximal number of surviving zeroes.

Theorem 1 *Either the number of surviving different zeroes (including* 0*) of* I *is less or equal then* $(\deg P + \deg Q)/3$*, or* P *is proportional to* Q*.*

Proof :

5 Verification of the main conjecture.

5.1 Note about rescaling of P and Q.

As it was shown in [2], it is possible, using rescaling $x \mapsto C_1 x$, $y \mapsto C_2 y$, to make the leading coefficients of P, Q beeing equal to any positive number. It can be done if $\deg Q \neq 2 \deg P$, but these cases will be not considered in this article. So we will use polynomials P and Q in the form with the leading coefficient equals either 1 or 2, i.e. if required we will be able to deal with polynomials in the form (for $W(x) = x(x-a)$):

$$R(x) = W^k W' + \beta_k + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0),$$

or

$$R(x) = W^k + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + ... + (\alpha_0 W' + \beta_0),$$

5.2 Verification of the main conjecture for the case deg P=3, deg Q=4

The goal of this subsection is to prove that in this case $I = \bigcup\limits_{k=1}^{\infty} I_k$ can not have zeroes, others then 0. Since in this case we can not represent $P(x) = \tilde{P}(W(x))$, where $W(x)$ is polynomial, accumulating common zeroes, the conjecture for this case is true.

1) From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 1 \ , \ \deg Q_1 = 2.$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W' + \alpha \ , \ Q_1 = W + \beta W' + \gamma.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that for all α, β, γ they can not have zeroes 0, a. It will complete verification of the main conjecture (4) for this case, since it is impossible to present $P = \tilde{P}(W(x))$.

3) We will calculate consecutively $\psi_k(a)$, using the following "Mathematica" program:

```
W=x(x-a);
W'=2x-a;
P=W*(W'+al);
Q=W*(W +bt*W' + ga);
p=D[P,x];
q=D[Q,x];
k0=0;
k1=1;
k2=-P;
k3=Integrate[-2k2*p-1k1*q,x];
k4=Integrate[-3k3*p-2k2*q,x];
k5=Integrate[-4k4*p-3k3*q,x];
k6=Integrate[-5k5*p-4k4*q,x];
k7=Integrate[-6k6*p-5k5*q,x];
k8=Integrate[-7k7*p-6k6*q,x];
x=a;
Print["k2=   ",Simplify[k2]];
Print["k3=   ",Simplify[k3]];
Print["k4=   ",Simplify[k4]];
Print["k5=   ",Simplify[k5]];
Print["k6=   ",Simplify[k6]];
Print["k7=   ",Simplify[k7]];
Print["k8=   ",Simplify[k8]];
```

Running this program, we obtain the following results:

$$\psi_4(a) = \frac{-\left(a^5\ (2\,a^2 + 7\,\alpha\,\beta - 7\,\gamma)\right)}{210},$$

$$\psi_5(a) = \frac{a^7\ \alpha\ (a^2 + 3\,\alpha\,\beta - 3\,\gamma)}{315}.$$

Since $a \neq 0$, we get

$$\alpha\beta - \gamma = -2a^2/7 \ , \ \alpha(\alpha\beta - \gamma + a^2/3) = 0.$$

It can be satisfied only if $\alpha = 0$, $\gamma = 2a^2/7$. Running the program for these values, we get the following conditions:

$$\psi_6(a) = \frac{a^{11}\ (13 - 21\,a^2)}{4414410},$$

$$\psi_7(a) = \frac{-2\,a^{13}\,\beta}{315315},$$

from which we obtain $\beta = 0$, $a = \pm\sqrt{13/21}$, and for them we get

$$\psi_8(a) = -3668/9,$$

i.e. we obtain contrudiction. The conjecture is verified.

Remark: We can do the same calculations for $P = \lambda P$, $Q = \mu Q$. For $\lambda \neq 0$, $\mu \neq 0$ from conditios $\psi_3(a) = \psi_4(a) = \psi_5(a) = 0$ we obtain exactly the same conditions on α, γ. Then

$$\psi_6(a) = \frac{a^{11}\,\lambda\,\mu\ (-21\,a^2\,\lambda^2 + 13\,\mu)}{4414410},$$

$$\psi_7(a) = \frac{-2\,a^{13}\,\beta\,\lambda^2\,\mu^2}{315315}.$$

From $\psi_7(a) = 0$ we get $\beta = 0$, so now the condition $\psi_8(a) = 0$ is

$$\psi_8(a) = \frac{-\,(a^{15}\,\lambda\,\mu\ (490\,a^4\,\lambda^4 - 2527\,a^2\,\lambda^2\,\mu + 969\,\mu^2))}{9980981010},$$

and the conditions $\psi_6(a) = 0$ and $\psi_8(a) = 0$ are again incompatible.

5.3 Verification of the main conjecture for the case deg P=4, deg Q=2

The goal of this subsection is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has zeroes 0 and a if and only if $Q(x) = W(x) = x(x-a)$, $P(x) = \tilde{P}(W(x))$.

Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = W,$$

where $W = x(x-a)$, $\deg P_1 = 2$ For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent $P_1 = W + \beta W' + \gamma$. Then computing the condition $0 = \psi_4(a) = a^5\beta/30$, we obtain $\beta = 0$, q.e.d.

Remark: The same answer we obtain if we replace P by λP, Q by μQ: $\psi_4(a) = a^5\beta\lambda\mu/30$.

5.4 Remark about resultants.

Resultants give us a very powerful tool for checking, if $n+1$ polynomials of n variables $P_i(x_1, \dots, x_n) \in C[x_1, \dots, x_n]$ do not have common zeroes.

Consider one example. Assume we are interested if polynomials $P(x,y)$, $Q(x,y)$, $R(x,y)$ have common zeroes. Compute $\mathbf{Resultant}[P,Q,x] = f_1(y)$, $\mathbf{Resultant}[R,Q,x] = f_2(y)$. If $\mathbf{Resultant}[f_1, f_2, y] \neq 0$, then P, Q, R do not have common zeroes.

Indeed, if there exists common zero of all polynomials (x_0, y_0), then $f_1(y_0) = f_2(y_0) = 0$, hence $\mathbf{Resultant}[f_1, f_2, y] = 0$, q.e.d.

The general construction is exactly the same.

5.5 Verification of the main conjecture for the case deg P=4, deg Q=3

The goal of this subsection is to prove that in this case $I = \bigcup\limits_{k=1}^{\infty} I_k$ can not have zeroes, others then 0. Since in this case we can not represent $Q(x) = \tilde{Q}(W(x))$, where $W(x)$ is polynomial, accumulating common zeroes, the conjecture for this case is true.

1) From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes 0, a. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 2 \ , \ \deg Q_1 = 1.$$

For such P, Q numbers 0, a will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W + \beta W' + \gamma \ , \ Q_1 = W' + \alpha.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that for all α, β, γ they can not have zeroes 0, a. It will complete verification of the main conjecture (4) for this case, since it is impossible to present $Q = \tilde{Q}(W(x))$.

3) We will calculate consecutively $\psi_k(a)$, using the "Mathematica" program, similar to above.
Running the program, we obtain the following results:

$$\psi_4(a) = \frac{a^5 \ (2\,a^2 + 7\,\alpha\,\beta - 7\,\gamma)}{210},$$

$$\psi_5(a) = \frac{a^7 \ (4\,a^4 + 11\,a^2 \ (\alpha\,\beta - 3\,\gamma) - 66 \ (\alpha\,\beta - \gamma)\ \gamma)}{6930}.$$

From the first equation $\alpha\beta = \gamma - 2/7a^2$, substituting into the second equation, we obtain $\gamma = -a^2/77$. So, these and only these conditions force the equations $\psi_4(a) = 0, \psi_5(a) = 0$ to be satisfied. Obviously β may not be equal to zero, so we can put $\alpha = -a^2/77\beta$. Running the program for these values, we obtain the following equations:

$$\psi_6(a) = \frac{a^{11} \ (1573 - 21\,a^4\,\beta + 2541\,a^2\,\beta^3)}{534143610\,\beta},$$

$$\psi_7(a) = \frac{2\,a^{13} \ (-63954\,a^2 + 819\,a^6\,\beta + 3009391\,\beta^2 - 112651\,a^4\,\beta^3)}{948906123165\,\beta},$$

$$\psi_8(a) = (a^{13}(315517059a^2 + 47966683149\beta^2 - 1036350a^{10}\beta^2 - 6465588052a^4\beta^3$$
$$+151367370a^8\beta^4 + 10626a^6\beta(7543 + 93170\beta^5)))/(20166152929502580\beta^2).$$

We obtain three polynomials of two variables a, β. Now according to subsection 5.4. we can compute

$$\mathbf{Resultant}[\psi_6(a), \psi_7(a), \beta] = 171355466545636153516888971819\ a^2$$

$$-55381482335935291356128\ a^{12} + 4434102226084608\ a^{22},$$

$$\mathbf{Resultant}[\psi_6(a), \psi_8(a), \beta] = Const(216908655616510696903575187607931 a^8$$

$$+458786962298721610776125188208a^{18} - 161604860797505145100608a^{28}$$

$$+14080788862156800a^{38}),$$

and computing resultant of the last two expressions (dividing by the proper power of a) we get nonzero number, q.e.d.

Remark: We can do the same calculations for $P = \lambda P$, $Q = \mu Q$. For $\lambda \neq 0$, $\mu \neq 0$ from conditios $\psi_3(a) = \psi_4(a) = \psi_5(a) = 0$ we obtain exactly the same conditions on α, γ. Then

$$\psi_6(a) = \frac{a^{11}\lambda\mu(-21a^4\beta\lambda^2 + 2541a^2\beta^3\lambda^2 + 1573\mu)}{534143610\beta},$$

$$\psi_7(a) = Const\frac{a^{13}\lambda^2\mu(819a^6\beta\lambda^2 - 112651a^4\beta^3\lambda^2 - 63954a^2\mu + 3009391\beta^2\mu)}{\beta^2},$$

$$\psi_8(a) = Const(-1036350a^{10}\beta^2\lambda^4 + 151367370a^8\beta^4\lambda^4 - 6465588052a^4\beta^3\lambda^2\mu$$

$$+315517059a^2\mu^2 + 47966683149\beta^2\mu^2 + 10626a^6((93170\beta^6\lambda^4 + 7543\beta\lambda^2\mu))/\beta^2,$$

and repeating the same computations with resultants, we obtain the same result.

5.6 Verification of the main conjecture for the case deg P=4, deg Q=4

The goal of this subsection is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has common zeroes if and only if either $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$, where

$W(x) = x(x-a)$, or P is proportional to Q.

1) Let P, Q be not proportional. From the theorem 4.1. we get the maximal number of surviving zeroes is 2. And one of them is necessarily 0.

2) Assume that I has zeroes $0, a$. Since zeroes of I should be also zeroes of P and Q , P and Q are necessary represented in the form (up to rescaling)

$$P = WP_1 \ , \ Q = WQ_1,$$

where

$$W = x(x-a) \ , \ \deg P_1 = 2 \ , \ \deg Q_1 = 2.$$

For such P, Q numbers $0, a$ will be common zeroes of ideals I_1, I_2, I_3. Now represent

$$P_1 = W + \gamma W' - \alpha \ , \ Q_1 = W + \delta W' - \beta.$$

Then we will directly calculate, using the "Mathematica" software, ideals I_4, I_5, I_6, I_7, I_8 and we will show that the only possibilities for I to have zeroes $0, a$ are either $\gamma = \delta = 0$ or $P = Q$. It will complete verification of the main conjecture (4) for this case.

3) We will calculate consecutively $\psi_k(a)$, using the "Mathematica" program, similar to above. Running the program, we obtain the following results:

$$\psi_4(a) = \frac{a^5\ (7\,\alpha\,\delta + 2\,a^2\ (\delta - \gamma) - 7\,\beta\,\gamma)}{210},$$

$$\psi_5(a) = \frac{a^7\ (4\,a^4\ (\delta - \gamma) + 66\,\alpha\ (\alpha\,\delta - \beta\,\gamma) + 11\,a^2\ (3\,\alpha\,\delta - 2\,\alpha\,\gamma - \beta\,\gamma))}{6930}.$$

Since $a \neq 0$, we get

$$\frac{2}{7}a^2(\delta - \gamma) = \beta\gamma - \alpha\delta,$$

$$(4a^4 + 22a^2\alpha)(\delta - \gamma) + (\alpha\delta - \beta\gamma)(66\alpha + 11a^2) = 0.$$

If $\delta = \gamma$, then from the first equation $\alpha = \beta$, and hence $P = Q$. So, $\delta \neq \gamma$ and dividing the second equation by $\delta - \gamma$, we obtain $\alpha = -3a^2/11$. Then $\delta = 77\beta\gamma/a^2 + 22\gamma$.

Running the program for these values, we get

$$\psi_6(a) = \frac{-\left(a^7 \left(3\,a^2 + 11\,\beta\right)\,\gamma\,\left(-4719\,a^2 + 3\,a^6 - 17303\,\beta - 363\,a^4\,\gamma^2\right)\right)}{10900890}.$$

If $\gamma = 0$, then $\delta = 0$, q.e.d.

If $\beta = -3a^2/11$, then $\beta = \alpha$, hence $\delta = \gamma$, and hence $P = Q$.

So,

$$\beta = \frac{3a^6 - 4719a^2 - 363a^4\gamma^2}{17303},$$

and running the program for these values (i.e. without α, δ, β), we get

$$\psi_7(a) = (2a^{15}(a-11\gamma)\gamma(a+11\gamma)(508079a^2 + 711a^6 - 61477559\gamma^2 - 629926a^4\gamma^2$$
$$+99309903a^2\gamma^4))/1452013567609605$$

$$\psi_8(a) = (a^{15}(a-11\gamma)\gamma(a+11\gamma)(-165436111269a^2 + 23749415118a^6 + 37532547a^{10}$$
$$+20017769463549\gamma^2 - 2920126191268a^4\gamma^2 - 28998322881a^8\gamma^2 - 64672793651466a^2\gamma^4$$
$$+4252091239473a^6\gamma^4 + 52235717949261a^4\gamma^6))/6303887863505747915\,40.$$

$$\psi_9(a) = -(a^{19}(a-11\gamma)\gamma(a+11\gamma)(-166460483475a^2 + 4591313298a^6 + 8200347a^{10}$$
$$+22896141543275\gamma^2 - 436487623572a^4\gamma^2 - 5747949999a^8\gamma^2 - 49500160259550a^2\gamma^4$$
$$+782017522209a^6\gamma^4 + 20215061125875a^4\gamma^6)/3467138324928161353470$$

If $\gamma = \pm a/11$, then $\beta = -3a^2/11$, so $\alpha = \beta$ and hence $\gamma = \delta$, so $P = Q$.

If $\gamma = 0$, then $\delta = 0$, q.e.d.

Otherwise we get 3 polynomials in two variables γ, a. Computing resultants, we get nonzero number, q.e.d. The conjecture for this case is completely verified.

REFERENCES :

1. N. Bautin , *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of the type focus or center* , Trans. Amer. Math. Soc. **100** (1954) , 1-19

2. M. Briskin, J.-P. Françoise, Y. Yomdin , *Center conditions, composition of polynomials and moments on algebraic curves* , in preparation

3. M. Briskin, J.-P. Françoise, Y. Yomdin , *Polynomial moments on algebraic curves and composition of polynomials* , in preparation

4. L. Cherkas , *Number of limit cycles of an autonomous second-order system* , Differentsial'nye uravneniya **12** (1976) , No.5 , 944-946

5. A. Gasull, J. Llibre , *Limit cycles for a class of Abel equations* , SIAM J.Math. Anal. **21** (1990) , No.5 , 1235-1244

6. A. Neto , *On the number of solutions of the equation* $\frac{dx}{dt} = \sum_{j=0}^{n} a_j(t)x^j$, $0 \le t \le 0$, *for which* $x(0) = x(1)$, Inventiones math. **59** (1980) , 67-76

7. H. Poincaré , *Mémoire sur les courbes définies par les équations différentielles* , 1881

8. H. Poincaré , *Mémoire sur les courbes définies par les équations différentielles* , 1885

9. D. Schlomiuk , *Algebraic particular integrals, integrability and the problem of the center* , Trans. Amer. Math. Soc. **338** (1993) , No.2 , 799-841

10. D. Zeilberger , *A holonomic systems approach to special functions identities* , J. of Comp. and Appl. Math. **32** (1990) , 321-368

11. D. Zeilberger , *Private communication*