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CENTER AND MOMENT CONDITIONS FOR RATIONAL ABEL EQUATION

M. BLINOV, N. ROYTVARF, AND Y. YOMDIN

ABSTRACT. We consider the Abel Equation $\frac{dy}{dz} = p(z)y^2 + q(z)y^3$ as an equation on the complex plane with p, q – rational functions. The center problem for this equation (which is closely related to the classical center problem for polynomial vector fields on the plane) is to find conditions on p and q under which all the solutions $y(z)$ are univalued functions along the circle $|z| = 1$. In [3] we have shown that this problem is closely related to Moment Problem, namely conditions on p, q for vanishing of all the moments $\int_{|z|=1} P^i Q^j dP$ with $P = \int p$, $Q = \int q$. We slightly generalize an approach and consider an arbitrary curve $\gamma \in \mathbb{C}$ in place of the unit circle $|z| = 1$. The aim of this paper is to give a simple and constructive description of rational functions P and Q satisfying Moment Condition along γ, and to show that Moment Condition implies Center Condition for P and Q – Laurent polynomials.

1. Introduction

We consider the classical Center-Focus Problem for homogeneous polynomial vector fields on the plane (see e.g [12]): let $F(x, y)$, $G(x, y)$ be polynomials in x, y of degree d. Consider the system of differential equations

$$\dot{x} = -y + F(x, y), \ \dot{y} = x + G(x, y) \tag{1.1}$$

A solution $x(t)$, $y(t)$ of (1.1) is closed if it is defined in the interval $[0, t_0]$ and $x(0) = x(t_0)$, $y(0) = y(t_0)$. The system has a center at the origin if all the solutions around zero are closed. The classical Center-Focus problem is to find conditions on F and G such that all the trajectories of this system are closed curves around the origin.

It was shown in [5] that one can reduce the system (1.1) with homogeneous F, G of degree d to the **trigonometric Abel equation**

$$\frac{d\rho}{d\theta} = p(\theta)\,\rho^2 + q(\theta)\,\rho^3, \ \theta \in [0, 2\pi], \tag{1.2}$$

where $p(\theta)$, $q(\theta)$ are polynomials in $\sin\theta$, $\cos\theta$ of degrees $d+1$, $2d+2$ respectively. Then (1.1) has a center if and only if (1.2) has all the solutions periodic on $[0, 2\pi]$, i.e. the solutions $\rho = \rho(\theta)$ satisfying $\rho(0) = \rho(2\pi)$. A natural modification of the

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classical center-focus problem is to find conditions on p and q such that (1.2) has a center.

One can rewrite the differential equation (1.2) in a complex form, expressing $\sin\theta$ and $\cos\theta$ through $z = e^{i\theta}$, i.e.

$$\cos\theta = \frac{z+z^{-1}}{2},\ \ \sin\theta = \frac{z-z^{-1}}{2i},\ \ \rho = y,$$

so one obtains p and q in the form of **Laurent polynomials** in z, and Abel differential equation is

$$\frac{dy}{dz} = p(z)\,y^2 + q(z)\,y^3$$

considered on the circle $z([0,2\pi]) = S^1$.

We generalize slightly the problem and consider Abel equation

$$\frac{dy}{dz} = p(z)\,y^2 + q(z)\,y^3 \tag{1.3}$$

with $p(z)$, $q(z)$ – **arbitrary rational functions of the complex variable** z. We assume from the very beginning that the indefinite integrals of p and q do not contain logarithmic terms, i.e. $p = \frac{dP}{dz}$, $q = \frac{dQ}{dz}$ for rational P and Q. Let γ be a closed curve in $\mathbb{C}$ not containing poles of P and Q.

Definition 1.1 (Center Condition)**.** (1.3) **has a center along** γ if an analytic continuation along γ of any solution of (1.3) does not ramify.

Consider now the infinitesimal version of this problem:

Definition 1.2 (Tangential center problem)**.** Consider a family $y(\epsilon, x)$ of solutions along a given closed curve γ of

$$\frac{dy}{dz} = p(z)\,y^2 + \epsilon q(z)\,y^3, \tag{1.4}$$

with $y(\epsilon, a) = y_a$ for any ϵ.
Find conditions on P, Q, under which (1.4) has an **infinitesimal (or a tangential) center**, i.e.
1) it has a center for $\epsilon = 0$
2) $\hat{y}(x) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} y(x,\epsilon)$ is a univalued function along γ.

The following theorem was proved originally in [2]:

Theorem 1.3. *The functions P and Q define a tangential center if and only if for all nonnegative k and any $a \in \gamma$ all the moments*

$$m_k(x) = \int_a^x P^k(t)dQ(t) \tag{1.5}$$

are univalued functions along the curve γ

A slight generalization of this approach leads to the following

Definition 1.4 (Moment Condition)**.** P and Q satisfy **Moment Condition along** γ, if $m_{ij} = \int_\gamma P^i\, Q^j\, dP = 0$ for all nonnegative i, j.

Note. The condition $m_{ij}=0$ for all nonnegative i, j is equivalent to the condition $\hat{m}_{ij}=\int_\gamma P^i\,Q^j\,dQ=0$ for all nonnegative i, j. Indeed,

$$d\hat{m}_{i,j}=Q^j d\left(\frac{P^{i+1}}{i+1}\right)=d\left(\frac{Q^jP^{i+1}}{i+1}\right)-\frac{P^{i+1}}{i+1}dQ^j=d\left(\frac{Q^jP^{i+1}}{i+1}\right)-\frac{j}{i+1}dm_{i+1,j-1}$$

We shall use notations $m_{ij}(P,Q,\gamma)$ and $\hat{m}_{ij}(P,Q,\gamma)$.

Moment Condition plays a central role in many questions of Complex Analysis in one and several complex variables. In classical theorems of Wermer ([1], [13], [14]) and Harvey-Lawson [6] it is shown that Moment Condition is equivalent to the fact that the curve $\Gamma=(P,Q)(\gamma)\subset\mathbb{C}^2$ bounds a compact complex one-dimensional chain in $\mathbb{C}^2$.

To formulate our results, we need to consider a composition representation of the functions P and Q.

Definition 1.5. If there is a rational function W (of degree greater than one) such that $P(z)=\tilde{P}(W(z))$, $Q(z)=\tilde{Q}(W(z))$, with some rational $\tilde{P}$ and $\tilde{Q}$, then W is called **a (right) composition common factor of P and Q.** If no such W exists, P and Q are called **relatively prime** (in composition sense).

We use classical theorems by Wermer and Harvey-Lawson and composition representation to produce our constructive description of rational functions, satisfying Moment Condition:

Theorem 1 *Let P, Q be relatively prime (in composition sense) rational functions. Then $m_{ij}(P,Q,\gamma)=0$ for all $i,j\geq 0$ if and only if all the poles of P and Q lie on one side of γ.*

Exact formulation of the notion "points lie on one side of a curve" is given in section 2 below, and all the proofs are given in section 3.

Consider now the case where P and Q are not assumed to be relatively prime. Let W be the Composition Greatest Common Factor of P and Q, i.e. $P=\tilde{P}(W)$, $Q=\tilde{Q}(W)$, with $\deg W>1$, and $\tilde{P}$, $\tilde{Q}$ – relatively prime rational functions.

Theorem 2 *For P, Q as above, $m_{ij}(P,Q,\gamma)=0$ for all $i,j\geq 0$ if and only if all the poles of $\tilde{P}$ and $\tilde{Q}$ lie on the same side of $W(\gamma)$.*

Using these theorems, we lift all the restrictions imposed on our result in [3], and prove

Theorem 3 *For P and Q – Laurent polynomials, vanishing of $m_{ij}(P,Q,S^1)$ implies Center Condition on S^1.*

Besides Theorem 3, there are some other special cases where Moment Condition implies Center one.

Theorem 4 *Let P and Q be relatively prime rational functions, and let γ be a simple closed curve (without self-intersections). Then Moments condition for P and Q implies center.*

FIGURE 1. Signed number of intersection points

Corollary *Let W be a composition Greatest Common Factor of rational functions P and Q. If γ and $W(\gamma)$ are simple closed curves, then Moment Condition implies Center Condition.*

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2. SOME PRELIMINARY CONSTRUCTIONS

Both the Center and the Moment Conditions depend only on a homotopy class of γ as a curve in $\mathbb{C}$ without poles of P, Q. Hence we can perturb γ to satisfy any "general position" requirement. In particular, we always assume below that γ is a smooth curve with transversal self-intersections.

It is also natural to consider P and Q as defined on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C}\bigcup\infty$, and γ as a curve in $\mathbb{C}P^1$.

Let $p_1, \dots, p_r$ be the poles of P and $q_1, \dots, q_s$ the poles of Q on $\mathbb{C}P^1$. We denote by Δ the set $\{p_1, \dots, p_r, q_1, \dots, q_s\}$.

The condition on γ, which is central in all the results below, is the following: γ is homologous to zero in $\mathbb{C}P^1 \setminus \Delta$. However, to treat this condition in a convenient way, we would like to give it another interpretation: all the poles of P and Q lie on "one side of γ". This last condition, being geometrically clear for simple curves γ, requires more accurate explanation for curves with self-intersections. We give this (certainly, not new, but essential for our presentation) explanation below, following closely [1], [13], [14].

Let γ subdivide $\mathbb{C}P^1$ into a finite number of simply-connected domains D_j, $j = 1, \dots, N$. Let us fix one of these domains, say D_i and let us consider the plane $U = \mathbb{C}P^1 \setminus \{s\}$, s – any point in D_i. Then we can define **the index of γ with respect to each of the domains** D_j, $\mu_{ij}(\gamma)$ (or, shortly, μ_j), as the rotation of the curve γ around any point in D_j (in the plane U).

More accurately, $\mu_{ij}(\gamma)$ can be defined in several equivalent ways:

a. As an image of γ in the homology group $H_1(U \setminus \{p\}) \cong \mathbb{Z}$, p – any point of D_j.

b. As the (signed) number of intersection points with γ of any path, joining s and p, p – any point of D_j (see figure 1).

FIGURE 2. Linking number of p and γ

c. As the linking number in U of the curve γ and the point $p \in D_j$. I.e. for any complex one-dimensional chain Z in U, such that $\gamma = \partial Z$, $\mu_{ij}(\gamma)$ is the (signed) number of intersections of Z with p (see figure 2).

Notice that always $\mu_{ii} = 0$.

Now, all the domains D_j with $\mu_{ij} = 0$ form "one side" of the curve γ. All the domains D_k with $\mu_{ik} \neq 0$ form "another side".

Definition 2.1. Points $p_1, \dots, p_r$ are **"on one side of γ"**, if any two of these points can be joined by a path, for which a signed sum of crossings with γ is equal to zero. Another equivalent definition is that after fixing a domain D_i, containing one of the points p_l, all these points are contained in domains D_j with $\mu_{ij} = 0$.

Notice that this definition is not symmetric: any two points in the domains D_j with $\mu_{ij} = 0$ can be joined by a path with crossing number with γ equal to zero. This is not true for the domains on the "other side" of γ.

Let $\tilde{\Delta} = \{x_1, \dots, x_m\}$ be a finite set in $\mathbb{C}P^1$.

Lemma 2.2. *A curve γ in $\mathbb{C}P^1 \setminus \tilde{\Delta}$ is homologous to zero in $\mathbb{C}P^1 \setminus \tilde{\Delta}$ if and only if all the points of $\tilde{\Delta}$ are on one side of γ.*

Proof: Assume that the curve γ bounds a complex one-dimensional chain $\tilde{Z}$ in $\mathbb{C}P^1 \setminus \tilde{\Delta}$. We fix one of the points p_i of $\tilde{\Delta}$ and the domain D_i, containing p_i. The chain $\tilde{Z}$ lies, in particular, in $U = \mathbb{C}P^1 \setminus \{p_i\}$ and can be used to compute the indices of the domains D_j, as defined above. Thus, for any domain D_j with $\mu_{ij} \neq 0$, $\tilde{Z}$ covers each of the points of D_j with a nonzero multiplicity μ_{ij}. But since $\tilde{Z} \subseteq \mathbb{C}P^1 \setminus \tilde{\Delta}$, no point of $\tilde{\Delta}$ can belong to such D_j.

Conversely, assume that all the points of $\tilde{\Delta}$ are on one side of γ. This means that all the points of $\tilde{\Delta}$ belong to the domains D_k with $\mu_{ik} = 0$. Therefore, each of the domains D_j with $\mu_{ij} \neq 0$ lies in $\mathbb{C}P^1 \setminus \tilde{\Delta}$. Define the complex one-dimensional chain $\tilde{Z}$ in $\mathbb{C}P^1 \setminus \tilde{\Delta}$ as follows: $\tilde{Z} = \sum_j \mu_{ij} \bar{D}_j$. Clearly, $\tilde{Z} \subseteq \mathbb{C}P^1 \setminus \tilde{\Delta}$. Indeed, the points of $\tilde{\Delta}$ can appear only in domains D_j with $\mu_{ij} = 0$.

Now, using a definition of μ_{ij} as the crossing number (definition b. above), we get immediately that $\gamma = \partial \tilde{Z}$. This completes the proof of the lemma.

Another technical ingredient we need is an interpretation of the composition (Definition 1.5) in terms of algebraic geometry of rational curves.

The following facts are very basic in algebraic geometry. We restate them for convenience of our presentation. For details we address the reader to any classical algebraic geometry text (e.g. [11], see also [3]).

Lemma 2.3. *The curve* $Y = \{(P(t), Q(t)), t \in \mathbb{C}\} \subseteq \mathbb{C}^2$ *for* **rational** P *and* Q *is an algebraic curve.*

Lüroth theorem. *Any subfield of a field of rational functions is generated by a rational function.*

Corollary 2.4. *There exist rational functions* $W(t)$, $\bar{P}(t)$, $\bar{Q}(t)$, *s.t.* $P(t) = \bar{P}(W(t))$, $Q(t) = \bar{Q}(W(t))$ *and the map*

$$\bar{\varphi} : \mathbb{C} \to Y, \quad z \mapsto (\bar{P}(z), \bar{Q}(z))$$

defines a birational isomorphism between $\mathbb{C}$ *and* Y. *In particular,* Y *is a rational curve.*

Definition 2.5. The **degree of a map** $\varphi = (P, Q) : \mathbb{C} \to Y$ is the degree of the algebraic extension $[\mathbb{C}(t) : \mathbb{C}(P, Q)]$.

Corollary 2.6. *If* $\deg \varphi = 1$, *then* φ *defines birational isomorphism.*

Definition 2.7. A rational function $W(t)$ is called **Composition Greatest Common Factor (CGCF)** of rational functions $P(t)$, $Q(t)$, if $W(t)$ is a common factor under composition of $P(t)$ and $Q(t)$, and if $\tilde{W}(t)$ is another common factor of $P(t)$ and $Q(t)$ under composition, then $W(t) = R(\tilde{W}(t))$ for a rational function R.

Corollary 2.8. *For any rational functions* $P(t)$, $Q(t)$ *their Composition Greatest Common Factor* $W(t)$ *exists and is given by corollary 2.4. CGCF is unique in the algebra of rational functions under compositions up to composition with invertible rational functions (i.e. functions of degree 1), i.e. two CGCF of a given function can be obtained each one from another by (right and left) compositions with linear functions.*

Thus, the "greatest common factor" W of P and Q in composition sense exists, and it can be found effectively, assuming that we can solve algebraic equations. Notice, however, that the algebra of compositions of rational functions is rather complicated (see [9], [10]).

Lemma 2.9. *1) For a rational map* $\varphi = (P, Q) : \mathbb{C} \to Y$ *the number of preimages of almost each point is equal to* $\deg \varphi$.
2) $[\mathbb{C}(t) : \mathbb{C}(W(t))] = \deg W(t)$.

Corollary 2.10. *The degree of the map* $\varphi = [P, Q]$ *is equal to the degree of the rational CGCF of* P *and* Q. *If* W *is CGCF of* P *and* Q, $P = \bar{P}(W)$, $Q = \bar{Q}(W)$, *then* $\bar{\varphi} : \mathbb{C} \to Y$, $z \mapsto (\bar{P}(z), \bar{Q}(z))$ *has degree 1.*

Thus, for P and Q relatively prime, a mapping $\varphi = (P, Q) : \mathbb{C} \longrightarrow Y \subseteq \mathbb{C}^2$ is a birational isomorphism. According to a general description of birational isomorphisms [11], we can give now a complete topological description of the affine curve Y. Indeed, for a generic point of Y, $\varphi^{-1}(y)$ consists of one point. There is a finite number of points $y_1, \ldots, y_l \in Y$, which have more than one preimage under φ. Let $\varphi^{-1}(y_m) = \{x_{m1}, \ldots, x_{mn_m}\}$.

Finally, each pole of P or Q in $\mathbb{C}P^1$ corresponds to a point of Y "at infinity". Denote, as above, these poles by $p_1, \ldots, p_r, q_1, \ldots, q_s$, respectively. We get the following result:

Theorem 2.11. *Affine curve $Y \subseteq \mathbb{C}^2$ is homeomorphic (under $\varphi = (P,Q)$) to $\mathbb{C}P^1 \setminus \{p_1,\dots,p_r,q_1,\dots,q_s\}$ with the points $\{x_{m1},\dots,x_{mn_m}\}$ glued together for each $m = 1,\dots,l$.*

3. Main results

Theorem 3.1. *Let P and Q be relatively prime in composition sense. Assume that one of the poles $p_1,\dots,p_r,q_1,\dots,q_s$ belongs to the domain D_i of the partition of $\mathbb{C}P^1$ by γ. Then $m_{kl}(P,Q,\gamma) = 0$ for all $k,l \geq 0$ if and only if all the poles $p_1,\dots,p_r,q_1,\dots,q_s$ of P and Q are on one side of γ, i.e. they belong to the domains D_j with the zero indexes μ_{ij}.*

We can give also a description of the complex one-dimensional chain Z in $\mathbb{C}^2$, bounded by $\Gamma = (P,Q)(\gamma)$, which exists by Wermer and Harvey-Lawson theorems. Remind that by Lemma 2.2, if all the poles $p_1,\dots,p_r,q_1,\dots,q_s$ of P, Q are on one side of γ, then γ bounds a complex one-dimensional chain $\tilde{Z}$ in $\mathbb{CP}^1 \setminus \{p_1,\dots,p_r,q_1,\dots,q_s\}$.

Theorem 3.2. *Let P and Q be relatively prime in composition sense. Then $\Gamma = (P,Q)(\gamma)$ bounds a complex one-dimensional chain $Z \subseteq Y \subseteq \mathbb{C}^2$, $Z = (P,Q)(\tilde{Z})$.*

The proof of these theorems will be given below.

Consider now the case where P and Q are not assumed to be relatively prime. Let W be the CGCF of P and Q, i.e. $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, with $\deg W > 1$, and $\tilde{P}$, $\tilde{Q}$ – relatively prime rational functions. Perturbing, if necessary, the curve γ, we can assume, that also the curve $W(\gamma)$ has only normal crossings.

Corollary 3.3. *For P, Q as above, $m_{ij}(P,Q,\gamma) = 0$ for all $i,j \geq 0$ if and only if all the poles of $\tilde{P}$ and $\tilde{Q}$ lie on the same side of $W(\gamma)$.*

Proof: By definition, $m_{ij}(P,Q,\gamma) = m_{ij}(\tilde{P},\tilde{Q},W(\gamma))$, and the result follows from the theorem 3.1.

For P, Q as above, theorem 3.2 can be reformulated accordingly.

Corollary 3.4. *The set M_d of all the couples (P,Q) of rational functions of degree d satisfying Moment Condition on γ consists of the open components formed by relatively prime P, Q, having all the poles "on one side" of γ, and of components of a smaller dimension, formed by some couples (P,Q) with a nontrivial CGCF.*

Proof: Follows directly from Theorem 3.1 and Corollary 3.3

We can give another description of the set M of all the couples of rational functions , satisfying Moment Condition on γ.

Corollary 3.5. *The set M consists of all the couples $(\tilde{P},\tilde{Q})$ having poles on one side of γ, , and of those couples (P,Q) which are obtained from the previous ones by composition $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$ with a nontrivial W.*

Now we consider the case of P, Q – Laurent polynomials.

Corollary 3.6. *The set of Laurent polynomials P, Q with $m_{ij}(P,Q,S^1) = 0$ consists of P, Q with poles "on one side of S^1" and of P, Q, allowing a composition* *representation $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, with W – Laurent polynomial, $\tilde{P}$ and $\tilde{Q}$ – usual (algebraic) polynomials.*

Proof: Follows from the result above, and from the fact proved in [3]: there are only two possible composition representation of Laurent polynomials:
a) $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, with $\tilde{P}$ and $\tilde{Q}$ – usual (algebraic) polynomials and W – Laurent polynomial. In this case all the moments vanish since $\tilde{P}$ and $\tilde{Q}$ have a pole only at infinity.
b) Another possible factorization is $P = \tilde{P}(z^m)$, $Q = \tilde{Q}(z^m)$, with $\tilde{P}$ and $\tilde{Q}$ – Laurent polynomials. But the change of variables $W = z^m$ preserves the property of the poles of P and Q to be on one side of S^1, QED

Theorem 3.7. *For P and Q – Laurent polynomials, vanishing of $m_{ij}(P, Q, S^1)$ implies Center Condition on S^1.*

Proof: By corollary 3.6, Moment Condition for P, Q implies either a composition representability $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$, with $\tilde{P}$, $\tilde{Q}$ usual polynomials, or a property that all the poles of P and Q are on the same side of S^1. In both cases the Abel equation $y' = p(z)\, y^2 + q(z)\, y^3$ has a center on S^1, since P and Q are regular (in the first case, after factorization by W) in a simply-connected domain, bounded by S^1, QED

The problem for rational functions beyond Laurent polynomials is that the condition "all the poles of P and Q are on the same side of γ" is equivalent to the fact that γ is *homologically trivial* in $\mathbb{C}P^1 \setminus \{p_1, \dots, p_r, q_1, \dots, q_s\}$. However, γ may be *homotopically nontrivial*.

Consequently, we can not conclude that the Abel equation (1.3) does not ramify along γ. It would be interesting to find specific examples of rational P and Q and γ, for which the Moment Condition is satisfied, while the Center Condition is not.

Besides Theorem 3.7, there are some other special cases where Moment Condition implies Center one.

Corollary 3.8. *Let P and Q be relatively prime rational functions and let γ be a simple closed curve (without self-intersections). Then Moments condition for P and Q implies center.*

Proof: By theorem 3.1 all the poles of P and Q are on the one side of a simple closed curve γ. Hence γ bounds in $\mathbb{C}P^1$ a simply-connected domain D, where both P and Q are regular. But then the solutions of (1.3) on D can not ramify, QED

Corollary 3.9. *Let W be a composition Greatest Common Factor of rational functions P and Q. If γ and $W(\gamma)$ are simple closed curves, then Moment Condition implies Center Condition.*

Proof: Follows from the previous result and corollary 3.3.

Remark. It is interesting to notice that in the question, studied in this paper, namely, relation between the Moment and the Center Conditions on closed curves γ, existence of a nontrivial common composition factor W of P and Q seems to complicate a picture. Indeed, for γ a simple closed curve. $W(\gamma)$ may have self-intersections, and thus topological problem mentioned above can arise. Another example is given by equation (1.3) with $P = \rho$ and $Q = \rho'$, where ρ is the Weierstrass function of a certain period lattice on $\mathbb{C}$. The study of this example has been started in [4]. It is shown there that for γ a small circle around 0 in $\mathbb{C}$, the Moment Condition is satisfied while the Center Condition is not. It is important to notice, that ρ and ρ', being double periodic on $\mathbb{C}$, have a composition factorization with $W : \mathbb{C} \longrightarrow T$ – a factorization of $\mathbb{C}$ by the periodic lattice. This is apparently in contrast with the situation for polynomial Moment and Center problems on the interval $[a, b]$. Here composition factorization for P and Q (indeed, with $W(a) = W(b)$) is equivalent to the Moment condition [8], and conjecturally is equivalent to the Center. Notice, however, that by [7] the "reduced" Moment condition $m_{i0}(P, Q, [a, b]) = 0$ for all $i > 0$ does not imply Center condition.

Proof of Theorems 3.1 and 3.2:

We have rational P and Q with no nontrivial composition factor. Let, as above, $p_1, \dots, p_r$ be the poles of P and $q_1, \dots, q_s$ be the poles of Q, $\Delta = \{p_1, \dots, p_r, q_1, \dots, q_s\}$ and let $y_1, \dots, y_l \in Y = \varphi(\mathbb{C})$, $\varphi = (P, Q)$ be multiple points of Y, i.e. the points, having more than one preimage under φ.

Perturbing γ slightly, we can assume the γ is smooth and real analytic. Indeed, we can represent γ by zeroes of a smooth function F with regular critical points. Approximating F by a polynomial, while preserving critical points and values, provides a required analytic approximation of γ. We can assume that γ does not pass through the preimages under φ of the points $y_1, \dots y_l$ and that γ does not pass through critical points of P and Q. Hence, denoting $\Gamma = \varphi(\gamma)$ the image curve of γ under $\varphi = (P, Q)$ we obtain that Γ is a smooth real analytic curve in Y with transversal self-intersections, and $\varphi : \gamma \longrightarrow \Gamma$ is a 1-1 analytic homeomorphism.

The Moment condition on γ is now written as

$$m_{ij} = \int_\Gamma x^i\, y^j\, dx = \hat{m}_{ij} = \int_\Gamma x^i\, y^j\, dy = 0$$

for all $i, j \geq 0$. Now we apply the theorem of Harvey and Lawson ([6], see also [13], [1]). It claims that any Γ in $\mathbb{C}^2$ that satisfies a moment condition, bounds a compact complex one-dimensional chain Z in $\mathbb{C}^2$.

Lemma 3.10. *$Z \subseteq Y$.*

Proof: Γ is a real analytic curve. Hence locally near Γ there is only one complex analytic curve, containing Γ, namely Y. (Otherwise two complex curves Y and $\tilde{Y} \neq Y$ would intersect on the real curve Γ.) Hence, near Γ, $Z \subseteq Y$. By analytic continuation, $Z \subseteq Y$ globally, QED

Corollary 3.11. *Γ is homological to zero in Y.*

Lemma 3.12. *γ is homological to zero in $\mathbb{C}P^1 \setminus \Delta$.*

Proof: By theorem 2.11, Y is obtained from $\mathbb{C}P^1 \setminus \Delta$ by gluing together preimages of each of the points $y_1, \dots, y_l \in Y$. But γ does not intersect these preimages, and gluing a finite number of points together does not influence the first homology group. Hence the result of the lemma follows from Corollary 3.11

This completes the proof of one direction of the theorem 3.1, since, as it was explained above, γ is homological to zero in $\mathbb{C}P^1 \setminus \Delta$ if and only if all the poles of P and Q lie on "one side of γ".

Another direction (if all the poles of P and Q are on one side of γ, then all the moments vanish) is almost immediate: indeed, by lemma 3.12, γ bounds a regular chain in $\mathbb{C}P^1 \setminus \Delta$. But then vanishing of the moments follows by Cauchy Theorem. Theorem 3.1 is proved.

The proof of theorem 3.2 is immediate, since $\varphi = (P,Q)$ is one to one on γ. Hence $\Gamma = (P,Q)(\gamma)$ bounds the complex one-dimensional chain $(P,Q)(\tilde{Z}) \subseteq Y \subseteq \mathbb{C}^2$, QED

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Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

Current address: Theoretical Biology and Biophysics Group, Los Alamos National Laboratory, Los Alamos NM 87545

E-mail address: mblinov@lanl.gov, blinov@wisdom.weizmann.ac.il

Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail address: nina@wisdom.weizmann.ac.il

Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail address: yomdin@wisdom.weizmann.ac.il