Local Center Conditions for Abel Equation and Cyclicity of its Zero Solution

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Abstract

An Abel differential equation $y' = p(x)y^2 + q(x)y^3$ is said to have a center at a pair of complex numbers (a, b) if $y(a) = y(b)$ for any its solution $y(x)$ (with the initial value $y(a)$ small enough). Let p, q be polynomials and let $P = \int p, Q = \int q$. P and Q satisfy "Polynomial Composition condition" if there exist polynomials $\tilde{P}$, $\tilde{Q}$ and W such that $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$, and $W(a) = W(b)$. The main result of this paper is that for a fixed polynomial p (satisfying some minor genericity restrictions) and for a fixed degree d of a polynomial q there exists $\epsilon(p, d) > 0$ such that for any polynomial q of degree d with the norm of q at most $\epsilon(p, d)$ the Abel equation above has a center if and only if the Polynomial Composition condition is satisfied. On this base we also provide an upper bound for the cyclicity of the zero solution of the Abel equation (i.e. for the maximal number of periodic solutions which can appear in a small perturbation of the zero solution).

This research was supported by the ISF, Grant No. 264/02, and by the BSF, Grant No. 2002243.

1 Introduction

Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y) \\ \dot{y} = x + G(x,y) \end{cases} \tag{1.1}$$

with $F(x,y)$ and $G(x,y)$ vanishing at the origin with their first derivatives. The system (1.1) has a center at the origin if all the solutions around zero are closed. The classical Center-Focus problem is to find conditions on F and G necessary and sufficient for system (1.1) to have a center at the origin.

This problem together with a closely related second part of Hilbert's 16-th problem (asking for the maximal possible number of isolated closed trajectories of (1.1) with $F(x,y)$ and $G(x,y)$ polynomials of a given degree) persists till now all the attacks. Many deep partial results have been obtained (see [4, 5, 22, 33]) but general center conditions are not known even for $F(x,y)$ and $G(x,y)$ polynomials of degree 3.

The classical approach to the Center-Focus problem is to analyze the conditions on the parameters of the system (1.1) provided by the vanishing of the first several "obstructions" to the existence of the center. If as a result one can show the existence of the "first integral" of (1.1) then the system has a center and no further analysis of the obstructions is necessary. The problem is that already for $F(x,y)$ and $G(x,y)$ polynomials in x and y of degree 3 the obstructions analyzed till now do not necessarily imply the existence of the first integral of any known type.

An alternative approach to both the Center-Focus and Hilbert 16-th problems is provided by the study of the perturbed integrable situations (in particular, perturbed Hamiltonian vector fields). See [4, 5, 22]. The investigation of the perturbation version (or of the infinitesimal version) of the above problems led to many important results, in particular, to a serious progress in understanding of the analytic structure of Abelian integrals (see [22] and references there).

In the present paper we consider a certain variant of the Center-Focus problem (closely related to the original one) – the Center-Focus problem for the Abel differential equation

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3. \tag{1.2}$$

This problem is to provide necessary and sufficient conditions on p, q and $a, b \in \mathbb{C}$ for all the solutions $y(x)$ of (1.2) to satisfy $y(a) = y(b)$. The Abel equation version of the Center-Focus problem has been studied in [1, 2, 3, 17, 18, 19, 23] (see also [34]) and in many other publications. As shown in [11] and in subsequent papers, it suggests important technical simplifications and it opens important relations with classical Analysis and Algebra. Still, the Abel equation versions of both the Center-Focus and Hilbert 16-th problems apparently reflect the main difficulties of the original classical ones.

The investigation of the infinitesimal Center-Focus problem for the Abel equation has been started in [11] and in subsequent publications of the same authors and in [7, 17] and others. It turned out to be essentially a certain problem in Analysis related to the classical Moment problem on one side and to the Composition algebra of univariate polynomials on the other. By now a reasonable understanding of the infinitesimal Center-Focus problem for the Abel equation has been achieved, especially after the recent results of [24, 25, 26, 27, 28].

The main goal of the present paper is to use the results of this infinitesimal analysis back in the Center-Focus problem itself (at least, locally). This is done by a comparison of the (nonlinear) "Center equations" with their linear parts.

Let us remind that as in the classical case, also for the Abel equation the Center conditions in the space of the parameters of the problem are given by *an infinite* set of the "obstructions" i.e. of certain polynomial equations on the parameters ("Center equations" – see Section 3 below). Hence, by Hilbert's finiteness theorem the Center conditions are in fact provided by a certain *finite* number N of the Center equations. Formally one can say that the Center-Focus problem is just to find this number N. More constructive approach is to understand the structure of the Center equations and on this base to produce meaningful necessary and sufficient conditions for the Center (obtaining on the way also a bound for N).

This last approach is taken in the present paper. The infinitesimal analysis of the Center-Focus problem for the Abel differential equation leads to the "Moment vanishing condition" and to the "Moment equations" which are the linear parts of the Center equations. The Moment equations imply (usually) the "Composition condition" which in our setting is the main (and the only) "Integrability condition". In particular, the Composition condition implies Center. Moreover, it implies the vanishing of each nonlinear term in the Center equations. We translate these analytic and algebraic information into the information about the algebro-geometric structure of the Center equations. Finally, the analysis of this algebro-geometric structure implies our main results: local Center conditions for the Abel equation, description of the Bautin ideal and upper bounds for the "cyclicity" of the zero solution (i.e. for the number of the periodic solutions which can appear in a small perturbation of the zero one).

2 Statement of the main results

Consider the Abel differential equation

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3. \tag{2.1}$$

with $p(x) = P'(x)$ and $q(x) = Q'(x)$ polynomials in $x \in \mathbb{C}$.

Definition 2.1 *Let a pair of complex numbers (a, b) be given. The solution $y(x)$ of (2.1) is called periodic at (a, b) if $y(a) = y(b)$. Equation (2.1) is said to have a center at (a, b) if any its solution (with the initial value $y(a)$ small enough) is periodic at (a, b) (or equivalently, if $y(a) = y(b)$ for any its solution $y(x)$ with the initial value $y(a)$ small enough).*

For small initial values $y(a)$ the solutions $y(x)$ of (2.1) are regular in any fixed disk around $0 \in \mathbb{C}$. Hence we do not need to specify in Definition 2.1 the continuation path from a to b for the solutions $y(x)$.

Definition 2.2 *Polynomials $P(x)$ and $Q(x)$ satisfy a Polynomial Composition condition (PCC) at $(a, b) \in \mathbb{C}$ if there exist polynomials $\tilde{P}(w)$, $\tilde{Q}(w)$*

and $W(x)$ with $W(a) = W(b)$ such that

$$P(x) = \tilde{P}(W(x)),\ Q(x) = \tilde{Q}(W(x)). \tag{2.2}$$

Polynomial Composition condition (PCC) for the polynomials P and Q implies that Abel equation (2.1) with $p(x) = P'(x)$ and $q(x) = Q'(x)$ has a center at a, b. This simple but basic fact follows via a change of variables $w = W(x)$ in (2.1), which closes the integration contour, while the coefficients of the transformed equation still remain polynomials. Hence for small initial values the solutions $\tilde{y}(w)$ of the transformed equation do not ramify. But the solutions $y(x)$ of (2.1) are expressed as $y(x) = \tilde{y}(W(x))$ and since $W(a) = W(b)$ we have $y(a) = y(b)$ for any solution $y(x)$ of (2.1). A proof of a similar implication for iterated integrals is given in Proposition 3.4 below. See also [11].

For given P, Q the condition (PCC) can be effectively verified by algebraic calculations.

A composition condition similar to (PCC) has been introduced for a trigonometric Abel equation in [1, 2]. The condition (PCC) has been introduced and intensively studied in [11, 12, 13, 14, 6, 7, 17, 35] There is a growing evidence supporting the major role played by the Polynomial Composition condition (and in general, by the polynomial Composition Algebra – see [29]) in the structure of the Center conditions for the polynomial Abel equation. In particular, we have no counterexamples to the following "Composition conjecture":

Composition conjecture. *The Abel equation on the interval $[a, b]$ with p, q polynomials has a center if and only if (PCC) holds for $P = \int p, Q = \int q$.*

This conjecture has been verified for small degrees of p and q and in many special cases in [11, 12, 13, 14, 6, 7, 17, 35].

In this paper we take a "non-symmetric" approach to the Center-Focus problem for the Abel equation. Namely, we assume the polynomial p in (2.1) to be fixed while only the degree d of q is fixed and the polynomial q is allowed to vary inside the space V_d of all the complex polynomials of degree at most d. We shall show that under certain genericity assumption on the fixed p *locally with respect to $q \in V_d$* the Center condition for (2.1) and the Composition condition (PCC) are equivalent.

The assumption on the fixed p which is central for the method used in the present paper, is given by Definition 2.3 below. Let us stress, however, that this restriction can be presumably eliminated by considering higher order perturbations.

Consider the following "one-sided" moments:

$$m_k = \int_a^b P^k(x)q(x)dx, \ k = 0, 1, \dots . \tag{2.3}$$

The vanishing of the moments m_k (which we shall call a "Moment condition") is also implied by the Composition condition (PCC) (by the same reasons as above and as in Proposition 3.4 below. See also [11]).

Notice that the validity of the Moment condition does not depend on the choice of the constant terms in P and Q. This is immediate for Q since only $q = Q'$ enters into the expression (2.3). The moments for $P + c$ are linearly expressed through the moments for P and vice versa, so this is true also for P. In what follows we shall mostly assume that $P(a) = Q(a) = 0$.

Definition 2.3 *A polynomial P is called "definite" (with respect to $a, b \in \mathbb{C}$) if for any polynomial q the vanishing of the one-sided moments m_k, $0 \le k < \infty$, implies (and hence is equivalent to) the Polynomial Composition condition (PCC) for P and $Q = \int q$.*

All polynomials P up to degree 5 are definite. In the space V_l of polynomials P of a fixed degree $l \geq 6$ non-definite polynomials belong to a certain proper algebraic subset. More specifically, all indecomposable P are definite (for every $a \neq b$), as well as all P with $P'(a) \neq 0$, $P'(b) \neq 0$. Chebyshev polynomial T_6 is not definite with respect to $a = -\sqrt{3}/2$, $b = \sqrt{3}/2$ ([24]). In the Addendum below we present a survey of the recent results of [8, 9, 10, 11, 12, 13, 17, 25, 26, 27, 31, 28] describing several classes of definite polynomials.

The following theorem is the first main result of this paper:

Theorem 2.1 *Let a polynomial $p = P'$ be fixed, with the polynomial P definite. Let the maximal degree d of the polynomials q be fixed. There exists $\epsilon = \epsilon(p, d, a, b) > 0$ depending only on p, a, b and d such that for any q of* *degree d with $\| Q \| \leq \epsilon$ the equation $y' = p(x)y^2 + q(x)y^3$ has a center at a, b if and only if P and $Q = \int q$ satisfy the Polynomial Composition condition (PCC).*

The proof of theorem 2.1 is given in Section 4 below.

Remark. In a recent paper [35] an interesting analysis of the Center-Focus problem for the Abel differential equation is presented, partly similar to our approach. In particular, theorem 5.6 of [35] is essentially a special case of our theorem 3.1 for p of degree 1. Although formally the statement of theorem 5.6 of [35] is weaker (it does not guaranty the uniformity of the "locality size" with respect to the polynomials q of a fixed degree) we believe that the proof of theorem 5.6 of [35] essentially provides the uniform bound.

The approach of the present paper allows us to compute also the local Bautin ideal I of (2.1) i.e. the ideal in the ring of holomorphic functions on the ball B_ϵ of radius ϵ in V_d generated by all the Taylor coefficients $v_k(q)$ of the Poincaré first return mapping G on a, b. (See Section 3 below for an accurate definition of G). Moreover, we can compute also the Bautin index $b(P, d, a, b)$ which is the minimal number of v_k which generate I.

In a similar way we can define the stabilization index of the Moment equations. More accurately, let us define the "Moment Bautin index" $N(P, d, a, b)$ as follows:

Definition 2.4 *For P a definite polynomial on a, b and for any natural d the Moment Bautin index $N(P, d, a, b)$ is the minimal number of the moments $m_k = \int_a^b P^k(x)q(x)dx$ whose vanishing implies for any $q \in V_d$ that the Polynomial Composition condition (PCC) is satisfied by P and $Q = \int q$.*

The existence (finiteness) of $N(P, d, a, b)$ follows from the stabilization of the decreasing sequence of linear subspaces L_j defined by the vanishing of the moments m_k, $0 \leq k < j$, in the space V_d. A natural conjecture is that always $N(P, d, a, b)$ depends only on the *degree* of P and on d.

The following theorem is the the second main result of this paper:

Theorem 2.2 *Let $p = P'$ be fixed, with P definite. The local Bautin ideal I of Abel equation (2.1) on a, b coincides with the ideal J generated by all the moments $m_i(q)$. It is in fact generated by $v_2, \ldots, v_{N+3}$ or by $m_0, m_1, \ldots, m_N$, where $N = N(P, d, a, b)$ is the Moment Bautin index. In particular, the Bautin index $b(P, d, a, b)$ is equal to $N(P, d, a, b) + 3$.*

Theorem 2.2 implies an explicit bound on the "cyclicity" of the zero solution $y(x) \equiv 0$ of the Abel equation (2.1) i.e. on the number of periodic (those with $y(a) = y(b)$) solutions $y(x)$ of (2.1) which can bifurcate from the zero solution.

Let a definite P and $\deg q = d$ be fixed and let $\epsilon = \epsilon(P, d) > 0$ be as defined in theorem 2.1. The following theorem is the third main result of this paper (it is also proved in Section 4 below):

Theorem 2.3 *There is $\delta = \delta(P, d) > 0$ such that for any q with $\| q \| \leq \epsilon/2$ the number of solutions y of the Abel equation (2.1) satisfying $y(a) = y(b)$ and $|y(a)| \leq \delta$ does not exceed $N(P, d, a, b) + 3$.*

3 Poincaré mapping and Center equations

The *Poincaré first return mapping $G(y)$ of Abel equation (2.1) at $a, b \in \mathbb{C}$* associates to each $y = y_a$ the value $G(y) = y(b)$ at the point b of the solution $y(x)$ of (2.1) satisfying $y(a) = y_a$ at the point a. For $y = y_a$ sufficiently small $G(y) = y(b)$ does not depend on the continuation path from a to b, so $G(y)$ is a regular function for y near zero and it is given by a convergent power series

$$G(y) = y + \sum_{k=2}^{\infty} v_k(\lambda, a, b) y^k, \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the (finite) set of the coefficients of p, q.

The solution $y(x)$ of (2.1) is periodic at (a, b) if and only if $G(y(a)) = y(a)$. The equation (2.1) has a center at (a, b) if and only if $G(y) \equiv y$. Therefore we get the following simple but basic fact:

Proposition 3.1 *Abel equation (2.1) has a center at a, b if and only if the infinite sequence of equations*

$$v_k(\lambda, a, b) = 0, \quad k = 2, \ldots \tag{3.2}$$

is satisfied.

We shall call equations (3.2) the center equation and the set $\mathcal{C}$ of the parameters λ satisfying (3.2) we call the Center set.

It is convenient to "free" the endpoint b in (3.1): if we denote by $G(y,x)$ the Poincaré first return mapping $G(y)$ at a,x we obtain the following convergent Taylor representation which can be used to express both the Poincaré mapping (as we fix x) and the solutions of (2.1) (as we fix y):

$$G(y,x) = y + \sum_{k=2}^{\infty} v_k(\lambda, a, x) y^k. \tag{3.3}$$

One can easily show (by substituting expansion (3.3) into equation (2.1)) that $v_k(x) = v_k(\lambda, a, x)$ satisfy the recurrence relation

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ v_n(0) = 0 \quad \text{and} \\ v_n'(x) = p(x) \displaystyle\sum_{i+j=n} v_i(x)v_j(x) + q(x) \sum_{i+j+k=n} v_i(x)v_j(x)v_k(x), \;\; n \geq 2. \end{cases} \tag{3.4}$$

An immediate consequence is the following:

Proposition 3.2 *The Taylor coefficients $v_k(\lambda, a, x)$ are polynomials in a, x and in λ. In particular, for $x = b$ the coefficients $v_k(\lambda) = v_k(\lambda, a, b)$ are polynomials in the parameters a, b, λ.*

Proof: This follows from the recurrence relation (3.4) via induction by k.

So in fact Center equations (3.2) are polynomial ones. By Hilbert's finiteness theorem the Center set is in fact defined by a finite subsystem of (3.2) and in particular it is an algebraic subset of the space of the parameters.

It was shown in [11] that the recurrence relation (3.4) can be linearized in the following sense: consider the inverse Poincaré mapping G^{-1} associating to the end value $y(x) = y_x$ of each solution y of (2.1) its initial value $y(a) = y_a$. We have a Taylor expansion

$$y_a = G^{-1}(y_x) = y_x + \sum_{k=2}^{\infty} \psi_k(\lambda, a, x) y_x^k. \tag{3.5}$$

In particular, for $x = b$ we get the inverse to the Poincaré mapping G at a, b. Hence the Center condition $y(a) \equiv y(b)$ is equivalent to another infinite system of polynomial in λ equations

$$\psi_k(\lambda, a, b) = \psi_k(\lambda) = 0, \quad k = 2, \ldots. \tag{3.6}$$

In fact, one can show (see [11]) that for each $k = 2, \ldots$ the ideals $I_k = \{v_2(\lambda), \ldots, v_k(\lambda)\}$ and $I'_k = \{\psi_2(\lambda), \ldots, \psi_k(\lambda)\}$ in the ring of polynomials in λ coincide.

It was shown in [11] that for a fixed λ the Taylor coefficients $\psi_k(x)=\psi_k(\lambda, a, x)$ satisfy a *linear* recurrence relation

$$\begin{cases} \psi_0(x) \equiv 0 \\ \psi_1(x) \equiv 1 \\ \psi_n(0) = 0 \quad \text{and} \\ \psi'_n(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \ n \geq 2. \end{cases} \tag{3.7}$$

Now one can see that each $\psi_k(\lambda, a, b) = \psi_k(\lambda)$ can be written as a sum of iterated integrals: each summand has the form $Const \cdot \int q \int p \ldots \int p \int q$ (the order and the number of the integrands p and q varies). More accurately, the iterated integrals entering the polynomials $\psi_k(\lambda)$ are given by

$$I_\alpha = \int_a^b h_{\alpha_1}(x_1)dx_1 (\int_a^{x_1} h_{\alpha_2}(x_2)dx_2 \ldots (\int_a^{x_{s-1}} h_{\alpha_s}(x_s)dx_s)) \ldots). \tag{3.8}$$

Here α are the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_s)$ with $\alpha_j = 1$ or 2, and $h_1 = p, h_2 = q$.

Formally integrating recurrence relation (3.7) we can obtain in a combinatorial way the "symbolic" expressions for ψ_k through the sums of the iterated integrals (3.8). The first few of these expressions for ψ_k are as follows:

$$
\begin{aligned}
\psi_0 &\equiv 0 \\
\psi_1 &\equiv 1 \\
\psi_2 &= -\int p = I_1 \\
\psi_3 &= 2\int p \int p - \int q = 2I_{11} - I_2 \\
\psi_4 &= -6\int p \int p \int p + 3\int p \int q + 2\int q \int p = -6I_{111} + 3I_{12} + 2I_{21} \\
\psi_5 &= 24I_{1111} - 12I_{112} - 8I_{121} - 6I_{211} + 3I_{22} \\
\psi_6 &= -120I_{11111} + 60I_{1112} + 40I_{1121} + 30I_{1211} - 15I_{122} \\
&+ 24I_{2111} - 12I_{212} - 8I_{221}
\end{aligned}
$$

The basic combinatorial structure of the "symbolic" expressions for ψ_k produced via the recurrence relation (3.7) is given by the following proposition:

Proposition 3.3 *For each $k \geq 2$ the Poincaré coefficient ψ_k is given as the integer linear combination of the iterated integrals of p and q:*

$$\psi_k = \Sigma\ n_\alpha I_\alpha, \tag{3.9}$$

with the sum running over all the multi-indices $\alpha = (\alpha_1, \ldots, \alpha_s)$ for which $\sum_1^s \alpha_j = k - 1$. The number of the terms in the expression for ψ_k is the $(k-1)$-th Fibonacci number. The integer coefficients n_α are given as the products

$$n_\alpha = (-1)^s \Pi_{r=1}^s (k - \Sigma_{j=1}^r \alpha_j). \tag{3.10}$$

Proof: By induction. Assuming that the result is true for $k < m$ we apply the recurrence relation (3.7) and represent the terms ψ_{k-1} and ψ_{k-2} according to the expression (3.9). Integrating the right hand side of (3.7) we obtain ψ_k as the integer sum of the new iterated integrals, each one containing exactly one integrand more than before the integration. These new iterated integrals can be naturally split into the two groups corresponding to the two terms on the right hand side of the recurrence relation (3.7). In the first group the new integrand on the left is p and in the second group it is q. Hence the multi-indices α in these two groups are mutually different and these multi-indices cover together all the α with $\sum_1^s \alpha_j = k$. This proves also that the number of the terms in the expression for ψ_k is the $(k-1)$-th Fibonacci number. Also the formula (3.10) for the coefficients n_α follows immediately by induction from (3.7).

Remark. Another derivation of the iterated integrals form of the Center equations has been obtained in [16] by a completely different method.

As it was mentioned above, we assume in this paper that in the Abel equation (2.1)

$$y' = p(x)y^2 + q(x)y^3,$$

the polynomial p is fixed while q is considered as a variable polynomial belonging to the space V_d of all the univariate polynomials of a given degree d. So let us denote by $\mu = (\mu_0, \dots, \mu_d)$ the coefficients of the polynomial q. The parameters μ form a part of the complete set λ of the parameters of Abel equation (2.1). In the setting where the polynomial p is fixed we can consider the expressions $\psi_k = \psi_k(\lambda)$ introduced above as the functions $\psi_k(\mu)$ of the variables μ only.

Corollary 3.1 *For each $k \geq 2$ the Poincaré coefficient $\psi_k(\mu)$ is a polynomial in μ of the degree $[\frac{k-1}{2}]$.*

Proof: The iterated integrals I_α are polynomials in μ of the degree equal to the number of the appearances of q in the integral. By Proposition 3.3 the maximal number of the appearances of q in the integrals of the sum (3.9) is equal to $[\frac{k-1}{2}]$.

Proposition 3.3 provides immediately also the following information about the structure of the polynomials $\psi_k(\mu)$:

Corollary 3.2 *For each $k \geq 0$ the term of the degree 0 in μ in the polynomial $\psi_k(\mu)$ is $I_{1\dots1}$ with the coefficient $(-1)^k k!$. The term of the degree 1 is given by the integer linear combination of the iterated integrals I_α with exactly one appearance of q.*

Explicit analysis of the symbolic expressions for ψ_k is not easy. Integration by parts can be used to simplify them but ultimately it leads to a "word problem" which has been analyzed only partly (and only for the recurrence relation (3.2)) in [18, 19, 3].

However, some iterated integrals above containing more than one appearance of both p and q cannot be reduced to the one-sided or double moments by "symbolic" operations (including integration by parts). This follows, in particular, from the example (given in [7]) of the Abel equation (2.1) with the coefficients p and q - elliptic functions, for which all the double moments vanish while the Center equations are not satisfied.

Below in this paper we always work with the Center equations given by the system (3.6). Assuming, as usual, that $P(a) = Q(a) = 0$ and simplifying the subsequent equations via the preceding ones we obtain the following explicit form for the first seven Center equations in (3.6): (see e.g. [14]):

$$
\begin{aligned}
0 &= \psi_2(b) = -P(b) \\
0 &= \psi_3(b) = -m_0 = -Q(b) \\
0 &= \psi_4(b) = -m_1 \\
0 &= \psi_5(b) = -m_2 \\
0 &= \psi_6(b) = -m_3 - \frac{1}{2}\int_a^b pQ^2 \\
0 &= \psi_7(b) = -m_4 - 2\int_a^b PpQ^2 \\
0 &= \psi_8(b) = -m_5 - \int_a^b \frac{1}{2}Q^3 p + 23P^3 Qq - 77\int_a^b P^2(t)q(t)dt \int_a^t Pq
\end{aligned}
$$

The terms m_k appearing in these equations are the one-sided moments (2.3): $m_k = \int_a^b P^k(x)q(x)dx$. Notice that the first of these equations implies $P(b) = P(a) = 0$ and the second one implies $Q(b) = Q(a) = 0$.

The form of these initial Center equations suggests some important general patterns which can be proved by a combination of the integration by parts and of some combinatorial analysis. In particular, the iterated integrals where the integrand q appears exactly once can be transformed via integration by parts to the moments form. By corollary (3.2) we see that the linear in q terms of the Center equations are indeed the moments m_k:

Theorem 3.1 *In each ψ_k, $k \geq 2$, the sum of all the iterated integrals containing exactly one integrand q (i.e. the linear in μ part of the polynomial $\psi_k(\mu)$) is equal to the $(k-3)$-d one-sided moment $m_{k-3} = \int_a^b P^{k-3}(x)q(x)dx$, taken with the coefficient -1.*

The proof of this result is given in [15] (see a remark after Theorem 4.2 there).

In this paper we use only one additional fact concerning the structure of the Center equations. It is given by the following proposition:

Proposition 3.4 *If the polynomials P and Q satisfy a Polynomial Composition condition (PCC) on a, b then for $p = P'$ and $q = Q'$ all the iterated integrals I_α on a, b vanish. In particular, (PCC) implies vanishing of each of the terms in the Center equations (3.6).*

Proof: Under the factorization $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ provided by the Polynomial Composition condition (PCC) we can make a change of the independent variable $x \to w = W(x)$ in the iterated integrals. We get

$$I_\alpha = \int_{W(a)}^{W(b)} h_{\alpha_1}(w_1)dw_1 \int_{W(a)}^{w_1} h_{\alpha_2}(w_2)dw_2 \dots \int_{W(a)}^{w_{s-1}} h_{\alpha_s}(w_s)dw_s. \qquad (3.11)$$

Here $h_{\alpha_j}(w) = \tilde{p}(w) = \tilde{P}'(w)$ for $\alpha_j = 1$ and $h_{\alpha_j}(w) = \tilde{q}(w) = \tilde{Q}'(w)$ for $\alpha_j = 2$. Now since $\tilde{P}'(w)$ and $\tilde{Q}'(w)$ are polynomials, all the subsequent integrands in (3.11) are polynomials. But by the conditions we have $W(a) = W(b)$ and the most exterior integral must be zero, being the integral of a certain polynomial over a closed contour.

4 Proof of Main Results

To prove the local coincidence of the Center and the Composition conditions we have to translate the information on the Center equations obtained in the previous section into the algebro-geometric properties of these equations. We use the following result which is essentially a version of the "Nakayama Lemma" in Commutative algebra (see for example [21], chapter 4, lemma 3.4) adapted to our situation:

Lemma 4.1 *Let $f_1, \dots, f_m$ be polynomials in n complex variables. Let $f_i = f_i^1 + f_i^2$, $i = 1, \dots, m$, with all the f_i^1 homogeneous of degree d_1 and f_i^2 having all the terms of degrees greater than d_1.*

Let $C = \{f_1 = 0, \dots, f_m = 0\}$, $C^1 = \{f_1^1 = 0, \dots, f_m^1 = 0\}$. We assume in addition that $f_1^1, \dots, f_m^1$ generate the ideal I_1 of the set C^1 and that each f_i^2 vanishes on C^1.

Then there exists $\epsilon > 0$ such that for the ball B_ϵ in $\mathbb{C}^m$,
1. $C \cap B_\epsilon = C^1 \cap B_\epsilon$.
2. In the ring of holomorphic functions on B_ϵ the ideals $I = \{f_1, \dots, f_m\}$ and $I_1 = \{f_1^1, \dots, f_m^1\}$ coincide.

Proof: Since f_i^2 vanish on C^1, they belong to the ideal I_1 of the set C^1. By assumption, I_1 is generated by f_i^1. Hence we have

$$f_i^2 = \sum_{j=1}^{m} a_{ij} f_j^1, \tag{4.1}$$

with certain polynomials a_{ij}.

We can assume that $a_{ij}(0) = 0$. Indeed, since f_j^1 are homogeneous of degree d_1, while all the terms in f_i^2 have degrees strictly greater than d_1, we can omit the free terms of a_{ij} in (4.1) and the equality still remains valid.

From $f_i = f_i^1 + f_i^2$ and from (4.1) we get

$$\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \tilde{A} \begin{pmatrix} f_1^1 \\ \vdots \\ f_m^1 \end{pmatrix}, \tag{4.2}$$

where $\tilde{A} = (a_{ij}) + Id$. Since $a_{ij}(0) = 0$, $\tilde{A}(0) = Id$, and hence $\tilde{A}$ is invertible in a neighborhood of the origin in $\mathbb{C}^m$, in particular, in a certain ball B_ϵ, $\epsilon > 0$. (Of course, this radius ϵ depends on the polynomials $f_1, \dots, f_m$ and on their decomposition $f_i = f_i^1 + f_i^2$).

We get

$$\begin{pmatrix} f_1^1 \\ \vdots \\ f_m^1 \end{pmatrix} = \tilde{A}^{-1} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \tag{4.3}$$

(4.2) and (4.3) prove both the conclusions of Lemma 4.1: coincidence of the ideals I and I_1 and of their zero sets C and C_1 inside B_ϵ. This completes the proof.

Remark Assumptions of lemma 4.1 concerning the degrees of f_i^1 and f_i^2, and the fact that f_i^1 generate the ideal of their zero set C^1 are essential as show the following examples:

Example 1. Let $f^1(x_1, x_2) = x_1, \; f^2(x_1, x_2) = (-1 + x_2)x_1$. Then $C_1 = \{x_1 = 0\}$, but $f = f_1^1 + f_2^2 = x_1 x_2$, and $C = \{x_1 = 0\} \cup \{x_2 = 0\}$. This example illustrates importance of "separation of degrees" of f^1 and f^2.

Example 2. Let $f = f^1 + f^2 = y^3 + y^2 x^2$. Here $C_1 = \{y = 0\}$, while $C = \{y^2(y + x^2) = 0\}$ has two components at 0. Here y^3 does not generate the ideal of $\{y = 0\}$.

Proof of Theorems 2.1 and 2.2. Let a definite polynomial $p = P'$ be fixed as well as the degree d of the polynomial q and the points $a, b \in \mathbb{C}$. As above, V_d denotes the space of complex polynomials q of degree d. We denote by $\mathcal{C} \in V_d$ the Center set of the Abel equation (2.1), i.e. the set of $q \in V_d$ for which (2.1) has a center. We shall also denote by $\mathcal{L} \subset V_d$ the composition linear subspace, consisting of those $q = Q' \in V_d$ for which P and Q satisfy the Polynomial Composition condition (PCC).

The Center set $\mathcal{C} \in V_d$ is defined in the space V_d by the infinite system of polynomial equations (3.6): $\psi_k(\mu) = 0, \; k = 2, \ldots$.

Let $N = N(P, d, a, b)$ be the Moment Bautin index of the definite polynomial P on a, b. We apply Lemma 4.1 to the first $N + 3$ equations of the system (3.6). So $f_i(\mu) = \psi_i(\mu)$ and $f_i^1(\mu)$ are chosen to be the linear parts $m_{i-3}(\mu)$ of $\psi_i(\mu)$ while $f_i^2(\mu)$ contain all the non-linear in μ terms of $\psi_i(\mu)$, $i = 2, \ldots, N + 3$. Clearly, in this case the assumptions of Lemma 4.1 concerning the degrees of f_i^1 and f_i^2 and the fact that f_i^1 generate the ideal of their zero set C^1 are satisfied.

By definition of the Moment Bautin index $N = N(P, d, a, b)$ the zero set C^1 of the equations $f_i^1(\mu) = m_{i-3}(\mu) = 0$ for $i = 2, \ldots, N + 3$, is the composition linear subspace $\mathcal{L} \subset V_d$. By Proposition 3.4 the nonlinear parts $f_i^2(\mu)$ vanish on $\mathcal{L}$. Hence the last condition of Lemma 4.1 is also satisfied. We conclude that there exists $\epsilon > 0$ such that for the ball B_ϵ in $V_d = \mathbb{C}^{d+1}$ the following is true:

1. $C_{N+3} \cap B_\epsilon = \mathcal{L} \cap B_\epsilon$, where C_{N+3} is the zero set of the first $N+3$ Center equations $\psi_k(\mu) = 0,\ k = 2, \ldots, N+3$.

2. In the ring of holomorphic functions on B_ϵ the ideals $I'_{N+3} = \{\psi_2, \ldots, \psi_{N+3}\}$ and $I^1_N = \{m_0, \ldots, m_N\}$ coincide.

Notice that the radius ϵ of the ball B_ϵ in V_d where the above conclusions are valid depends only on the equations $\psi_k(\mu) = 0,\ k = 2, \ldots, N$ and on their decomposition into the linear and the nonlinear parts. In other words, $\epsilon = \epsilon(P, d, a, b)$ depends only on the fixed definite polynomial P, the degree d, and the points a, b.

It remains to notice that for each $k \geq N+4$ the polynomial $\psi_k(\mu)$ belongs to the ideal $I^1_N = \{m_0, \ldots, m_N\}$ and hence also to $I'_{N+3} = \{\psi_2, \ldots, \psi_{N+3}\}$. Indeed, by Proposition 3.4 $\psi_k(\mu)$ vanishes on the Composition subspace $\mathcal{L}$. Since I^1_N is the ideal of $\mathcal{L}$ we obtain $\psi_k(\mu) \in I^1_N = I'_{N+3}$. Therefore the ideal I'_{N+3} coincides with the local Bautin ideal $I = \{\psi_2, \ldots, \psi_{N+3}, \psi_{N+4}, \ldots\}$. In particular, this implies that the Bautin index $b(P, d, a, b)$ is at most $N+3$. On the other hand, $b(P, d, a, b)$ cannot be smaller than $N+3$. Indeed, if the ideal $I = I'_{N+3}$ were generated by a smaller number than $N+3$ of the polynomials ψ_j we could invert the proof of Lemma 4.1 and to conclude that a smaller number than N of the moments m_j generate the ideal of $\mathcal{L}$ – a contradiction with the definition of the Moment Bautin index. Therefore the Bautin index $b(P, d, a, b)$ is equal to the Moment Bautin index $N(P, d, a, b)$ plus 3.

From the equality of the local ideals $I = I'_{N+3} = I^1_N$ we get also the coincidence of their zero sets inside the ball B_ϵ:

$$C_{N+3} \cap B_\epsilon = \mathcal{C} \cap B_\epsilon = \mathcal{L} \cap B_\epsilon,$$

where as above $\mathcal{C}$ is the Center set of (2.1) i.e. the set of zeroes of the Bautin ideal I. This completes the proof of Theorem 2.1 and of Theorem 2.2.

Remark. In fact, each of the polynomials $\psi_k(\mu)$ belongs to the ideal generated by the moments $m_1, \ldots, m_N$ in the *global* ring of polynomials in μ.

However, the inverse inclusion is valid in only in the domain where we can invert the corresponding matrix $\tilde{A}$ (in particular, in B_ϵ). In geometric terms, the result of Theorem 2.1 does not exclude a possibility that the Center set $\mathcal{C}$ of (2.1) contains other components besides the Composition set $\mathcal{L}$. However, these components may appear only "far away" from the origin.

Proof of Theorem 2.3. This theorem follows directly from Theorem 2.2 and Theorem 2.3.9 of [20]. Formally the results of [20] are stated for the Bautin ideal and the Bautin index defined in the global ring of polynomials in μ, while Theorem 2.2 concerns the Bautin index defined in the ring of functions on the ball $B_r \subset V_d$. However, for $\mu \in B_{r/2}$ all the estimates of [20] remain valid via the analytic version of the "Effective division theorem".

Remark The bounds of [20] are usually far from being realistic, mostly because of the "worst case" estimates used in Hironaka's effective division algorithm. We believe that for many analytic families arising in relation to algebraic differential equations (including the most mysterious one - the Poincaré mapping) the "blind" application of the division algorithm can be replaced by a detailed study of the algebraic propertied of the Taylor coefficients $a_k(\lambda)$. In particular, for the "Moment generating function" $H(y) = \int_a^b \frac{q(x)dx}{1-yP(x)}$ this was done in [15]. Also in the situation considered in the present paper one can replace a general division algorithm by an improved version of the "linear Division Theorem" of [15] combined with an accurate computation of the Moments Bautin index $N(P,d,a,b)$, with an estimate of the "non-degeneracy" of the moment equations, and with a bounding of the norm of the Center equations $\psi_k(\mu) = 0$. We plan to present these results separately.

5 Computing the Moment Bautin index in examples

For P of degree two a convenient method for the analysis of the one-sided moments has been suggested in [6]. It is based on a representation of Q via the basis of the ring of polynomials of x considered as a module over the polynomials of P. In this basis (and for P of degree two) the moment equations get a very simple form, and the matrix representation of these equations can be explicitly analyzed. As a result we can compute explicitly the Moment Bautin index $N(P,d,a,b)$ for P of degree two with respect to the two zeroes $a,b \in \mathbb{C}$ of P. We present this computation below. However, for higher degrees of P the matrices become much more complicated and only partial results can be obtained by this method.

In [8, 9, 12, 13, 14] an algebraic method for the analysis of the moments vanishing has been developed. This method introduces a rather delicate algebraic techniques which relate moments of different orders. Recently this method has been extended in [10] to produce quantitative information on moments "near-vanishing". In particular, the following result is obtained in [10]: for a given P and $q = Q'$ define the "moment polynomials" $m_k(x)$ by

$$m_k(x) = \int_a^x P^k(t)q(t)dt, \tag{5.1}$$

where a is one of the roots of P. Notice that the zero moment polynomial $m_0(x)$ is equal to $Q(x)$.

We define the "Generalized Moment vanishing" condition requiring that all the moment polynomials $m_k(x)$ vanish at certain fixed zeroes $x_1, ..., x_l$ of P, $x_1 = a$. The "Generalized Composition condition" is that $P(x) = \tilde{P}(W(x))$ and $Q(x) = \tilde{Q}(W(x))$ with $\tilde{P}(0) = 0, \tilde{Q}(0) = 0$ and with $W(x)$ vanishing at $x_1, ..., x_l$.

Theorem 5.1 *Let P be a polynomial of degree m. Fix l different zeroes $x_1, ..., x_l$, $x_1 = a$, of P with $2l \geq m+1$. Then for any Q of degree d vanishing of $N(P,d) = [(d-l)/(2l-m)] + 1$ moments $m_k(x)$ at all the points $x_1, ..., x_l$ implies Generalized Composition condition (which in this case takes the following form:*

$$P(x) = W^n(x), \ Q(x) = \tilde{Q}(W(x)),$$

where $W(x)$ is a certain polynomial vanishing at all the roots of P.

If not all the above moments vanish then the deviation of Q from the Generalized Composition condition can be estimated through the maximum of the absolute values of $m_k(x_j)$, $k = 0, 1, ..., N(P,d)$, $j = 1, ..., l$.

In particular, this theorem allows us to bound explicitly the Moment Bautin index $N(P,d,a,b)$ for any P of degree three with respect to any two its zeroes $a, b \in \mathbb{C}$: it does not exceed $d-1$. It provides also an explicit bound for the "locality size" in the above computations. We plan to present the corresponding results separately.

So let us fix a polynomial $P(x)$ of degree 2. We can always assume that one of the roots of $P(x)$ is zero and so $P(x) = x(x-b)$, $b \neq 0$. The following theorem provides a description of the Moments vanishing conditions for any polynomial Q:

Theorem 5.2 *Let $P(x) = x(x-b)$, $b \neq 0$. Let Q be a polynomial of degree d and let $\nu = [d/2]+1$. Let $m_j = m_j(b)$ be defined by (5.1). Then*

1) If for some k we have $m_k = m_{k+1} = \ldots = m_{k+\nu} = 0$, then there exists a polynomial $\tilde{Q}$ such that $Q(x) = \tilde{Q}(P(x))$. In this case all the moments m_j vanish.

2) For any number $r < \nu$ of the equations $m_{j_s} = 0, s = 1, \ldots, r$ (not necessarily consecutive) there exists a polynomial Q of degree d for which all these equations are satisfied, and which cannot be represented as $Q(x) = \tilde{Q}(P(x))$.

In particular, the Moment Bautin index $N(P, d, 0, b)$ is equal to $[d/2]+1$.

Proof: The proof of Theorem 5.2 consists of several steps. First of all, using the polynomial P and its derivative P' we can construct a basis for the space of all the polynomials in x of a given degree. Indeed, the polynomials $P(x)^k$ have the degree $2k$, and the polynomials $P(x)^k\ P'(x)$ have the degree $2k+1$, respectively. Therefore all these polynomials are linearly independent and $P(x)^k$, $P(x)^k\ P'(x)$, $k = 0, 1, \ldots, l$, form the basis of of the space V_{2l+1} of all the polynomials $r(x)$ of the degree at most $2l+1$. The same polynomials except $P(x)^l\ P'(x)$ form the basis of V_{2l}.

Thus any polynomial $r(x)$ of the degree $2l+1$ can be uniquely written in the form

$$r = P^l(\alpha_l P' + \beta_l) + P^{l-1}(\alpha_{l-1} P' + \beta_{l-1}) + \ldots + (\alpha_0 P' + \beta_0). \qquad (5.2)$$

We have the following simple proposition:

Proposition 5.1 *The polynomial $R(x)$ has a form $R(x) = \tilde{R}(P(x))$ if and only if in the representation (5.2) for its derivative $r(x) = R'(x)$ all the coefficients $\beta_j = 0$ for $j = 0, \ldots, l$.*

Proof: If all the coefficients $\beta_j = 0$ for $j = 0, \ldots, l$ then integrating the representation (5.2) we obtain

$$R = (\frac{\alpha_l}{l+1})P^{l+1} + (\frac{\alpha_{l-1}}{l})P^l + \ldots + \alpha_0 P + \delta, \tag{5.3}$$

or $R(x) = \tilde{R}(P(x))$ with $\tilde{R}(z) = (\frac{\alpha_l}{l+1})z^{l+1} + \ldots + \delta$. Conversely, if $R(x) = \tilde{R}(P(x))$ then differentiating this expression we get a representation (5.2) for $r(x) = R'(x)$ where only the terms $P(x)^k\ P'(x)$ have nonzero coefficients.

Remark. In fact the ring $\mathcal{R}$ of polynomials in x is a module over the polynomials in P. The representation (5.2) shows that $\mathcal{R}$ as a module has exactly two generators: 1 and P'.

We return now to the proof of Theorem 5.2. Represent $q(x) \in V_d$ according to (5.2):

$$q(x) = P^m(\alpha_m P' + \beta_m) + P^{m-1}(\alpha_{m-1}P' + \beta_{m-1}) + ... + (\alpha_0 P' + \beta_0). \tag{5.4}$$

For any j we get

$$m_j = \int_0^b P^j q = \beta_m \int_0^b P^{m+j} + \beta_{m-1}\int_0^a P^{m+j-1} + \ldots + \beta_0 \int_0^b P^j, \tag{5.5}$$

since all the integrals containing P' vanish on the interval $[0, b]$ with the endpoints – zeroes of P. Defining the constants ω_j by $\omega_j = \displaystyle\int_0^b P(t)^j dt$, we get the following proposition:

Proposition 5.2 *In the coordinates α_k, β_k of (5.4) the equation $m_j = 0$ takes the form $\sum_{k=0}^m \omega_{j+k}\beta_k = 0$.*

Therefore, the equations $m_{j_1} = 0, \ldots, m_{j_n} = 0$ can be rewritten as the system

$$\begin{cases} \beta_m\omega_{m+j_1} & + & \beta_{m-1}\omega_{m+j_1-1} & + & \cdots & + & \beta_0\omega_{j_1} & = 0 \\ \beta_m\omega_{m+j_2} & + & \beta_{m-1}\omega_{m+j_2-1} & + & \cdots & + & \beta_0\omega_{j_2} & = 0 \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ \beta_m\omega_{m+j_{n+1}} & + & \beta_{m-1}\omega_{m+j_{n+1}-1} & + & \cdots & + & \beta_0\omega_{j_{n+1}} & = 0 \end{cases}$$

i.e.

$$\begin{pmatrix} \omega_{j_1} & \omega_{j_1+1} & \cdots & \omega_{j_1+m} \\ \omega_{j_2} & \omega_{j_2+1} & \cdots & \omega_{j_2+m} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{j_{n+1}} & \omega_{j_{n+1}+1} & \cdots & \omega_{j_{n+1}+m} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = 0.$$

For $n < m$ this system always has a nonzero solution. According to Proposition 5.1, for the corresponding polynomial $q \in V_d$ its primitive $Q = \int q$ cannot be represented as $Q(x) = \tilde{Q}(P(x))$. This proves the second part of Theorem 5.1.

To prove the first part of this theorem, let us denote by $D_{k,m}$ the determinant of the system of m *consecutive moments equations* i.e. the determinant

$$D_{k,m} = \det \begin{vmatrix} \omega_k & \omega_{k+1} & \cdots & \omega_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k+m} & \omega_{k+m+1} & \cdots & \omega_{k+2m} \end{vmatrix}.$$

Proposition 5.3 *For any natural k and m $D_{k,m} \neq 0$.*

Proof: Consider the scalar product $< f, g >= < f, g >_k = \int_0^a P(x)^k f(x) g(x) dx$ on the space of square integrable functions on $[0, b]$. (Without loss of generality we can assume $b < 0$ and so $P(x)$ is positive on $[0, b]$). Then

$$D_{k,m} = \det \begin{vmatrix} < P^0, P^0 > & < P^0, P^1 > & \cdots & < P^0, P^m > \\ < P^1, P^0 > & < P^1, P^1 > & \cdots & < P^1, P^m > \\ \vdots & \vdots & \ddots & \vdots \\ < P^m, P^0 > & < P^m, P^1 > & \cdots & < P^m, P^m > \end{vmatrix}$$

is a Gramm determinant of the set of linearly independent vectors $P^j(x) = (x(x-b))^j$, hence it is non-zero.

Now we can complete the proof of Theorem 5.2. By Proposition 5.3 the vanishing of m consecutive moments $m_k, m_{k+1}, \ldots, m_{k+m}$ implies that in the representation (5.4) of q all the coefficients β_j must be zero. By Proposition 5.1 this implies that $Q(x) = \tilde{Q}(P(x))$ which in turn implies the vanishing of all the moments m_j.

Remark. One can find explicitly the values of ω_k: we have

$$\omega_k = \int_0^b (x(x-b))^k dx = (-1)^k \frac{b^{2k+1} k!}{2^k (2k+1)!!}.$$

Estimating the volume spanned by the vectors $P^j(x)$ with respect to the scalar product $< f, g >_k$ we can get an explicit lower bound for the determinants $D_{k,m}$. We do not use these bounds in the present paper. In [15] some explicit bounds for the locality size and for periodic solutions of the Abel equation are obtained via the methods of [10].

6 Addendum: Definite polynomials

At present we do not have a complete description of definite polynomials. However, some rather wide classes of definite polynomials have been recently specified. Let us present shortly these classes and give an outline of some relations between them.

1. Simple end-points. If a and b are simple zeroes of the polynomial P then it is definite on $[a, b]$. This was shown in [17]. Other proofs can be found in [25, 28].

2. Indecomposability. Any indecomposable polynomial P (i.e. not possessing a nontrivial composition representation $P(x) = R(S(x))$ with the degrees of both polynomials R and S greater than 1) is definite on each interval $[a, b]$ with $P(a) = P(b)$. This fact is proved in [25]. In particular, each P of a prime degree is indecomposable and hence definite on each interval $[a, b]$ with $P(a) = P(b)$.

3. Simple zeroes not at the end-points. If all the zeroes of P, except possibly a, b are simple then the polynomial P is definite on $[a, b]$. This is shown in [27]. Another result of [27] is the following: if for any critical value c of P except possibly 0 the preimage $P^{-1}(c)$ contains exactly one critical point of P then P is definite on any $[a, b]$ with $P(a) = P(b) = 0$.

4. Real polynomials. Let P be a polynomial with real coefficients and let $a, b \in \mathbb{R}$. If all the *real zeroes of* P *in the open interval* (a, b) are simple then P is definite on $[a, b]$. This is shown in [28]. The techniques presented in [14], Section 4.2, allow one to prove the following result: let P be a polynomial with real coefficients and let $a, b \in \mathbb{R}$, $P(a) = P(b) = 0$. Assume that the multiplicities of each of the roots of P on the closed interval $[a, b]$ are odd. Then P is definite on $[a, b]$. (We plan to present the proof of this and other results for real polynomials separately).

5. Geometry of $P([a, b])$. The following results are proved in [28]:
Let $P(x)$ be a complex polynomial, $P(a) = P(b) = 0$. Assume that there exists a path $\Gamma \subset \mathbb{C}$ joining a and b such that the curve $\gamma = P(\Gamma)$ has only transversal self-intersections and that the point $0 = P(a) = P(b)$ is on the boundary of the exterior domain with respect to the closed curve γ. Then $P(x)$ is definite on $[a, b]$.

The following criterion can be explicitly verified in many important examples: Let P be a complex polynomial with $P(a) = P(b) = 0$, $a, b \in \mathbb{C}$. Let Γ be a piecewise-analytic curve in $\mathbb{C}$ joining a and b and let $\gamma = P(\Gamma)$. Assume that the open part $\gamma \setminus 0$ is contained in an open Ω with piecewise-analytic boundary and assume that 0 belongs to the exterior boundary of Ω. Then $P(x)$ is definite on $[a, b]$. In particular, this happens for any $P(x) = (x - a)(b - x)P_1(x)$, with $P_1(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \in \mathbb{C}$, if the convex hull CH of the coefficients a_k does not contain $0 \in \mathbb{C}$. (In his case Ω is some open cone $\alpha < Arg(z) < \beta$ containing the closed cone with the vertex at $0 \in \mathbb{C}$ generated by CH).

6. Recursive representation. As it was mentioned above, in [8, 9, 10, 12, 13, 14] an algebraic method for the analysis of the moments vanishing has been developed, which relates between them the moments of different orders. This method provides a general setting of the Moment problem which is in some aspects more natural that the one used in the present paper: the moments vanish not only at two points a, b but at a possibly larger number of the roots of P. Theorem 5.1 above presents some initial results in this direction. The notion of a definite polynomial can be extended to this general setting and some classes of "generalized definite polynomials" can be described. We plan to present these results separately. As the setting of the present paper is concerned, the Recursive representation method shows that each polynomial P of degree at most three is definite. Another conclusion is that a polynomial P of degree d having at a, b zeroes of a total multiplicity d or $d-1$ is definite (this follows also from p.3).

7. Bernstein Classes. The method of [31] combines the study of the integration operator with the "Bernstein Classes" approach of [32]. The result is that any P is definite on $[a, b]$ for a, b different from a certain finite number of points (which in general do not coincide with the critical values of P as in pp. 1-3 above).

It was mentioned above that the Chebyshev polynomial $T_6(x)$ is not definite on the interval $[\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. The results of pp. 1-7 show that each polynomial P of degree at most 5 is definite. Indeed, for $\deg P \leq 2$ this follows from p.1 (and this was shown also in Section 5 above). For $\deg P = 2, 3, 5$ the polynomial P is indecomposable and hence definite on any a, b with $P(a) = P(b)$. Finally, for $\deg P = 4$ either the roots of P at a, b are simple or the remaining roots are simple. Hence, P is definite by p.1 or p.3, respectively. This fact can be proved also by the method of p.6.

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