Some Linear Algebra

Basics

- A VECTOR is a quantity with a specified magnitude and direction
  - Vectors can exist in multidimensional space, with each element of the vector representing a quantity in a different dimension. When there is only one dimension, the vector is said to be a *scalar*
- A MATRIX is a rectangular array of quantities
  - The rows or columns of a matrix are vectors

Linear Algebra

- Vector and matrix structures are made to exist in a mathematically constructed space, that is, a vector space. They inherit specific algebraic properties from the vector space that make them powerful mathematical objects.

- In our discussion to follow, we mean *matrix* and *vector* to represent not only a notational structure, but the structure imbued with the appropriate algebraic properties. We say that the study of these objects is ***linear algebra***

Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ is a matrix

Scalar operations on a matrix are done element-wise. For example, if k is a scalar, for scalar multiplication

$$k \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}$$

Matrix addition (and subtraction) is straightforward: element by element

Matrix multiplication is a whole different kettle of fish. It is NOT element by element multiplication. Instead it is a convolution, multiplying and mixing rows and columns of both matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$$

M1 M2

Note the process:
First row, first element of M1 **times** first row, first element of M2
PLUS
First row, second element of M1 **times** second row, first element of M2

Then
First row, first element of M1 **times** first row, second element of M2
PLUS
First row, second element of M1 **times** second row, second element of M2

etc for the second row of M1

check out http://www.purplemath.com/modules/mtrxmult.htm for an easy to understand demo

Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2$

Expanding,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

Note:

- The number of rows in the first matrix must equal the number of columns in the second
- Matrix multiplication is not commutative: AB $\neq$BA necessarily

Matrix Multiplication of a Vector

Matrices can multiply vectors (provided the orders of each are the same) using the same rules as matrix multiplication

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ is a matrix $$v = \begin{bmatrix} x \\ y \end{bmatrix}$$ is a vector

$$M\mathbf{v} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Note that the result is a vector. The operation of the matrix on the vector, which ‘transforms’ the vector to a new direction, can be though of as a *rotation*

Identity Matrix

The identity matrix is a square matrix that has 1's along its main diagonal and has zero for all other elements. Here it is shown for order 3, but the concept is the same for a square matrix of any order

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An important property is that if it multiplies any other square matrix M of the same order, the product is still M

Inverse Matrix

If M is an invertible matrix, then its inverse M^{-1} is a square matrix of the same order such that $MM^{-1}=I$

Matrix Transpose

Rows and columns are swapped. Need not be a square matrix

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Division does not exist, *per se*. Instead, the dividend is multiplied by the inverse of the divisor

$$\frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{\begin{bmatrix} w & x \\ y & z \end{bmatrix}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{-1}$$

Determinants
For square matrices only

For a 2x2 matrix, it's easy

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det M = |M| = ad - bc$$

Determinants

For a higher order matrices, it's not quite so easy….

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The strategy for a 3x3 matrix is to subdivide the matrix into determinants of 2x2 matrices called minors relating to the elements of the first row. The signs alternate. For example

$$\det M = \begin{bmatrix} a & & \\ & \begin{vmatrix} e & f \\ h & i \end{vmatrix} \end{bmatrix} - \begin{bmatrix} & b & \\ \begin{vmatrix} d & & f \\ g & & i \end{vmatrix} \end{bmatrix} + \begin{bmatrix} & & c \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \end{bmatrix}$$

det M= a(ei-hf)-b(di-gf)+c(dh-ge)=aei+bgf+cdh-ahf-bdi-cge

The above strategy generalizes to square matrices of order > 3

Determinants..who cares?

Why would we care about determinants in this course?

- The determinant must be non-zero to have a unique inverse
- When matrices represent linear systems, the determinant must be zero in order to have a non-trivial solution

Determinants in finding an inverse

Finding the inverse (if it exists) of a square matrix of order 2 can easily be done with nifty little formula that invokes the determinant:

- Swap the elements on the main diagonal
- Reverse the signs of the 2 off-diagonal elements
- Divide by the determinant

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formulas like this for higher order matrices exist but are ugly. Instead it is customary to use a row-specific technique called the *Gauss-Jordan* method. The matrix is adjoined to the identity matrix. Row operations (add, subtract, scalar divides and multiplies) are carried out, row by row, until the M part is transformed into the identity. The originally I part will become the inverse of M

$$M : I \longrightarrow I : M^{-1}$$

Eigenvalues and Eigenvectors

$$Let\ A\ be\ an\ n \times n\ matrix.$$

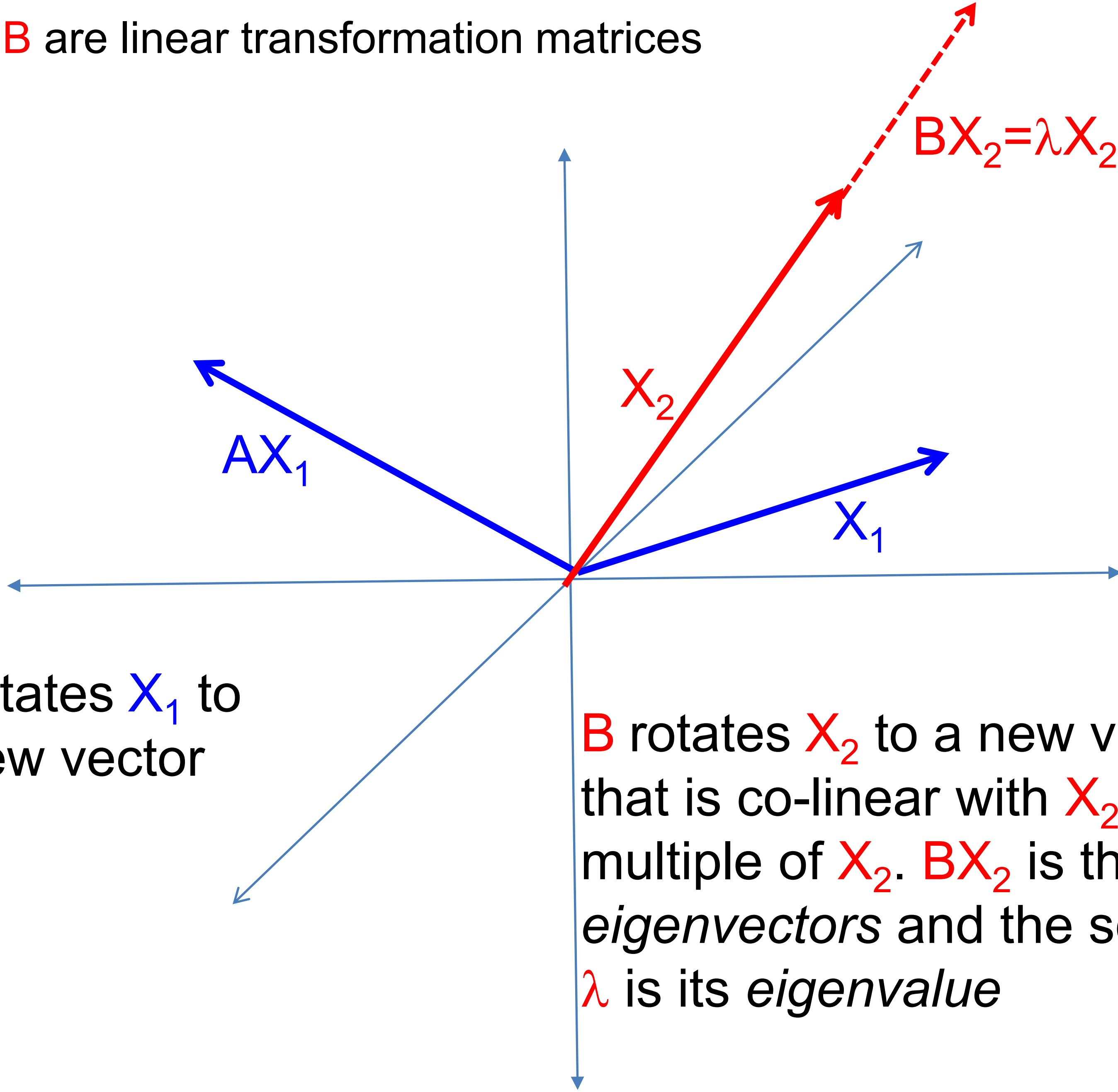
$$\lambda\ is\ an\ eigenvalue\ of\ A\ if\ \exists\ non-zero\ vector\ X\ in\ \Re^n such\ that$$

$$AX = \lambda X$$

This means that operating (rotating) some vector X by the matrix A yields the very same vector back again, multiplied by a real number, λ, an eigenvalue

If X is a non-zero vector satisfying the above, X is an eigenvector. There can be more than one solution; there are as many solutions (eigenvectors) as the order (n) of the matrix, although the solutions may not be distinct.

A and B are linear transformation matrices



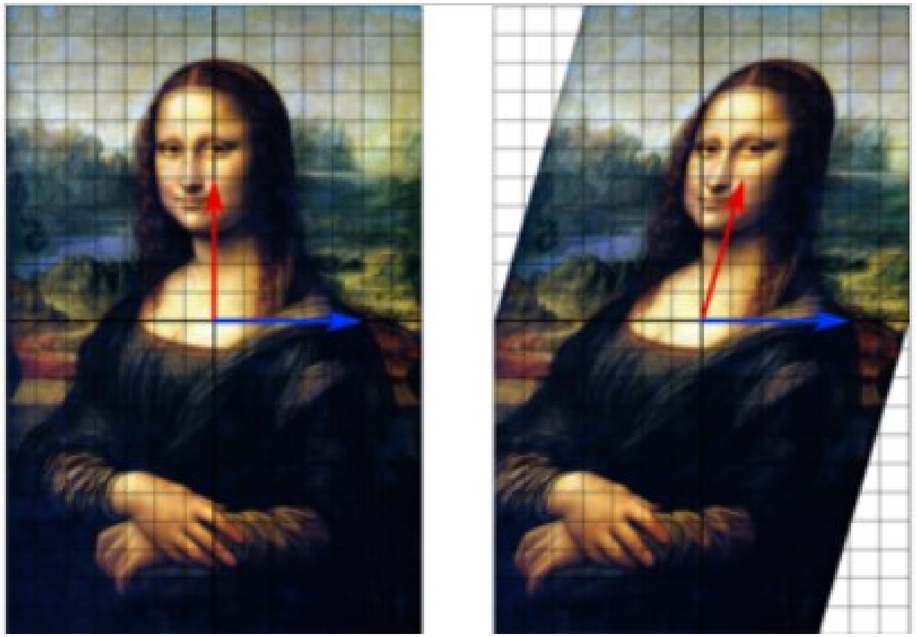
A rotates X_1 to a new vector AX_1

B rotates X_2 to a new vector BX_2 that is co-linear with X_2; it is a scalar multiple of X_2. BX_2 is the one of the *eigenvectors* and the scalar multiple λ is its *eigenvalue*

When you do this rotation, you need to know two things about where you end up:

1. What vectors emerge from the rotation (the eigenvectors)

2. What scale factors emerge from the rotation (the eigenvalues)



Given **A**X=λX, which represents a linear system of equations.

Then, re-writing this:
AX-λIX=0 where I is the identity matrix

For this linear system to have a non-trivial solution, its determinant must be zero.

So, $$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$ for a system of rank 2

Solving, we get $(a_{11}-\lambda)(a_{22}-\lambda)-(a_{12})(a_{21})=0$ (called the characteristic equation) resulting in a quadratic (in a matrix of rank 2) equation in λ. This produces two values for λ in a system of rank 2, λ_1 and λ_2.

These λ are the characteristic roots, or ***eigenvalues***, of the system. There will be an ***eigenvector*** corresponding to each eigenvalue

Example

$$For\ A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\ find$$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where λ is the eigenvalue

As a linear system,

$$\begin{aligned} x_1 + x_2 &= \lambda x_1 \\ -2x_1 + 4x_2 &= \lambda x_2 \end{aligned}$$

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 4) + 2 = 0$$

$$\lambda_1 = 2, \lambda_2 = 3$$

Eigenvalues may not be distinct

Finding the eigenvectors for λ_1=2

$$\begin{bmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 - 1 & -1 \\ 2 & 2 - 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Consider the top row 1 v_1-1v_2

Consider the 2nd row 2 v_1-2v_2

$$\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

In either case the eigenvector corresponding to eigenvalue λ_1 is any real number in the first element v_1 and its negative as the second element v_2