Jean-Baptiste Joseph Fourier 1768 –1830

Théorie analytique de la chaleur 1822

A function of any variable can be represented by a series of sines of the multiples of that variable.

The Fourier Series

- Represents any <u>periodic</u> function as a trigonometric series
- Trigonometric series converges if indeed the function is periodic
- Represents the function as:
  - A series of sines and cosines of some fundamental frequency and its harmonics

    or *Really ,really key alternative* ↓

  - A series of complex numbers that can be resolved into a series of Magnitudes and corresponding phase angles of some fundamental frequency and its harmonics

The Discrete Fourier Series

IF *f(x) is any* periodic waveform, then according to Fourier's Theorem

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Notice the ∞ in the summation; Fourier proved that the periodic waveform converges to the above expression in the limit.

BUT WATCH OUT : f(x) must be periodic over infinite time. If it is truncated, it is not periodic.

The Fourier Transform

A bilinear mapping for a periodic function (usually in the time or space domain) into the frequency domain

TIME DOMAIN

FREQUENCY DOMAIN

Fourier Transform

Periodic function

Series of complex coefficients at each harmonic

Inverse Fourier Transform

Euler's formula, fundamental to all of this:

$$\cos(\theta) - i\sin(\theta) = e^{-i\theta}$$

Proof:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x. \end{aligned}$$

$$\text{for } i^n \left| \begin{array}{ll} n = 0 & 1 \\ n = 1 & i \\ n = 2 & -1 \\ n = 3 & -i \end{array} \right.$$

The continuous Fourier Transform

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Frequency Domain Time Domain

The inverse Fourier transform

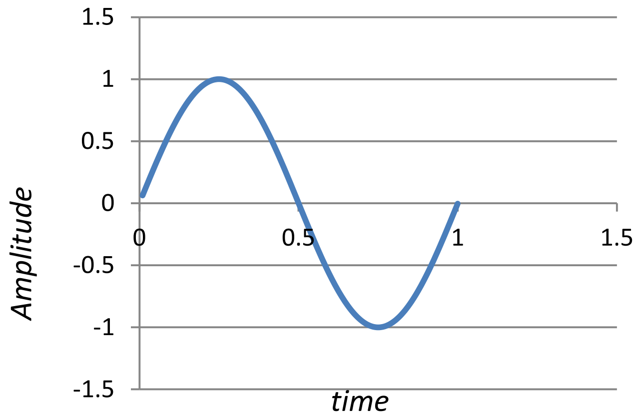
Note: no -

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} dt$$

Time Domain

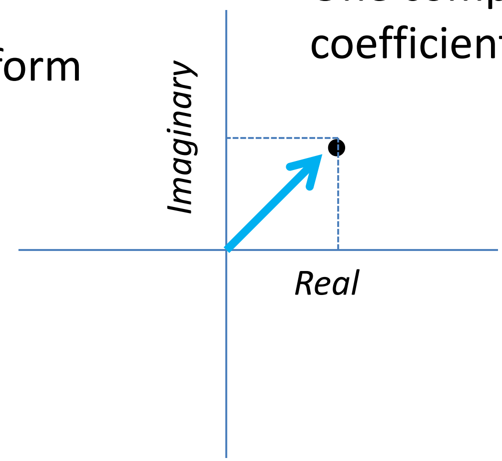
Frequency Domain

Very Simple Example



A simple periodic function

Fourier Transform



One complex Fourier coefficient at 1Hz

Representation of the simple periodic function in the frequency domain. The coefficient is a complex number. Magnitude is shown here in blue as

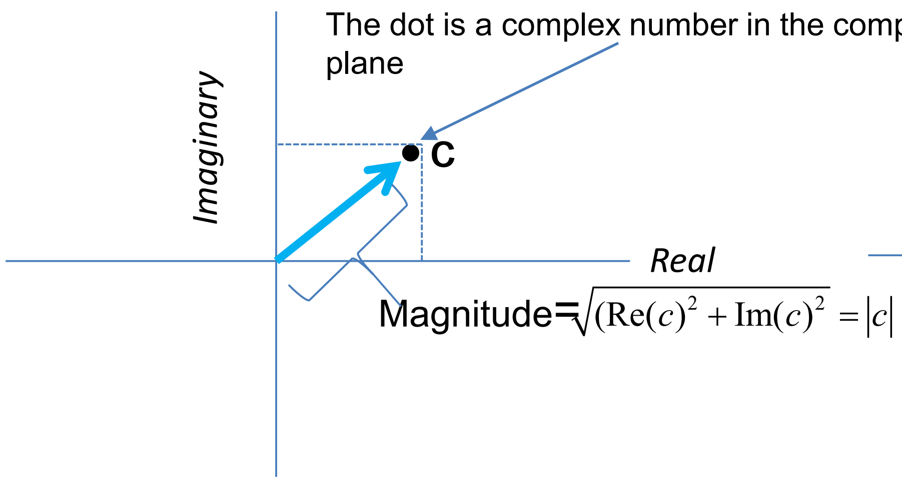
$$\sqrt{(\mathrm{Re}^2+\mathrm{Im}^2}$$

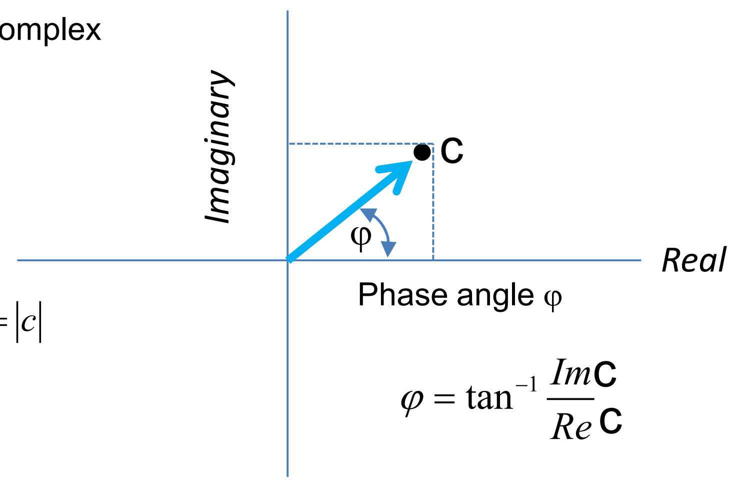
The Fast Fourier Transform

- The Fast Fourier Transform (FFT) was conceived Gauss in 1805, implemented for a computer in 1965 by Cooley (IBM) and Tukey (Bell Labs)
- In the 60's, processors were slow and memory was scarce , slow and very expensive
- The algorithm is a discrete transform, and requires that the number of sampling points (harmonics or multiples of the fundamental frequency) be an integral power of 2
- Exploiting the mathematics at these points, and manipulating the binary representation, the complex coefficients are generated efficiently
- The coefficients are accurate at the same data points as the FDFT
- The algorithm runs $\mathcal{O}(N\log_2 N)$, a dramatic speedup from N^2, the efficiency of the finite discrete FT, particularly with a large number of data points. For 2048 data points, 4,194,304 calculations are needed for the FDFT *vis à vis* 22,528 for the FFT

Magnitude, phase angle, and power

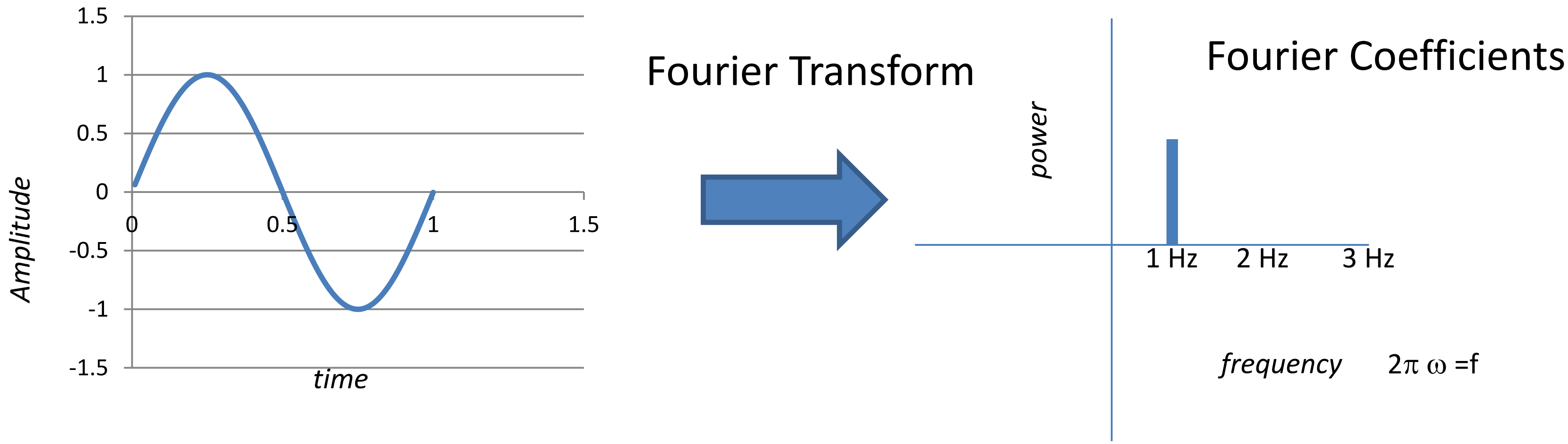
The Fourier coefficient is a complex number. We normally don't think in terms of complex numbers and the complex plane. An intuitive way to think of the complex number is a real magnitude associated with a phase angle





Very often, for a number of compelling reasons, it is convenient to think in terms of power, the square of the magnitude $|c|^2$

Very Simple Example



A simple periodic function

Representation of the simple periodic function in the frequency domain.
For display convenience, power, instead of magnitude, is shown here because power is a real number

About power

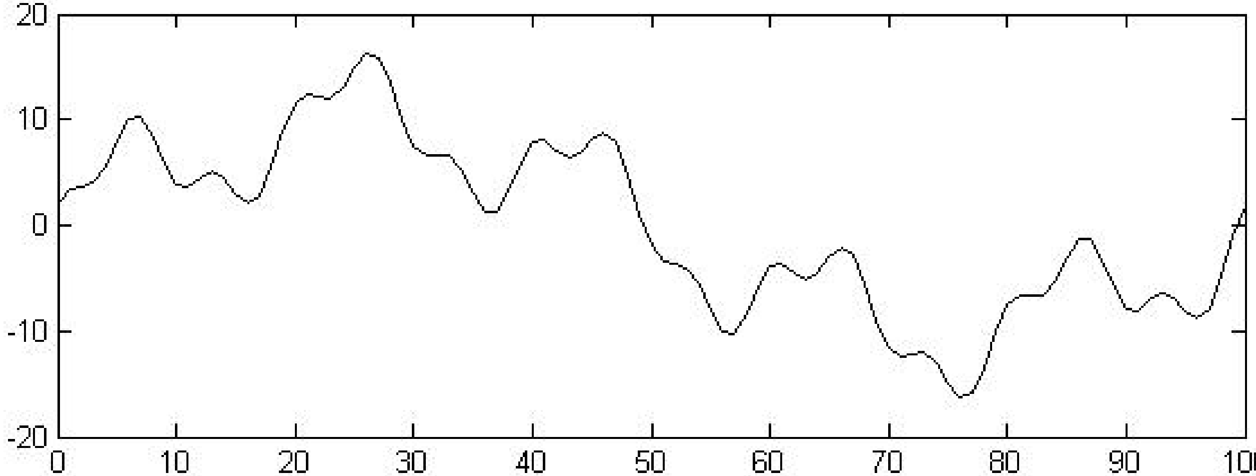
- Power is a real number, not complex
- Power of each coefficient is, mathematically, the square of the magnitude of each coefficient.
- The ensemble of powers of ordered coefficients is called the *power spectrum*, or *power spectral density (PSD)*

Power is calculated as Re^2+Im^2

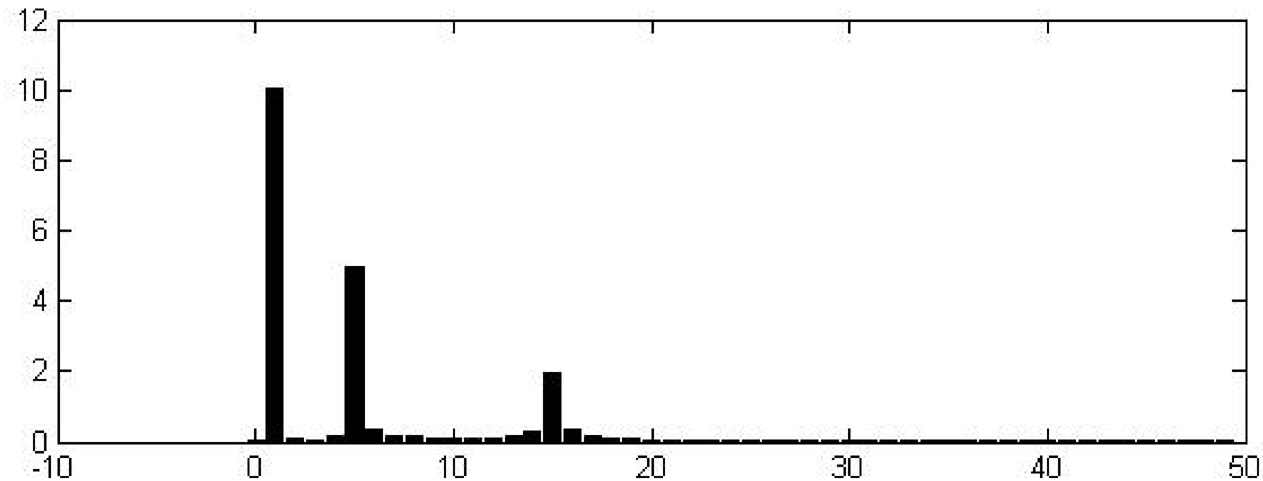
Alternatively, it is sometimes convenient to calculate power directly from the complex coefficient times its complex conjugate: $c \times \overline{c}$

Note that there is significant information loss when taking the power or magnitude of a complex coefficient; all the phase information is lost and inverse transformation is not possible.

Practical uses of the power spectrum:
Teasing meaning from confusing data



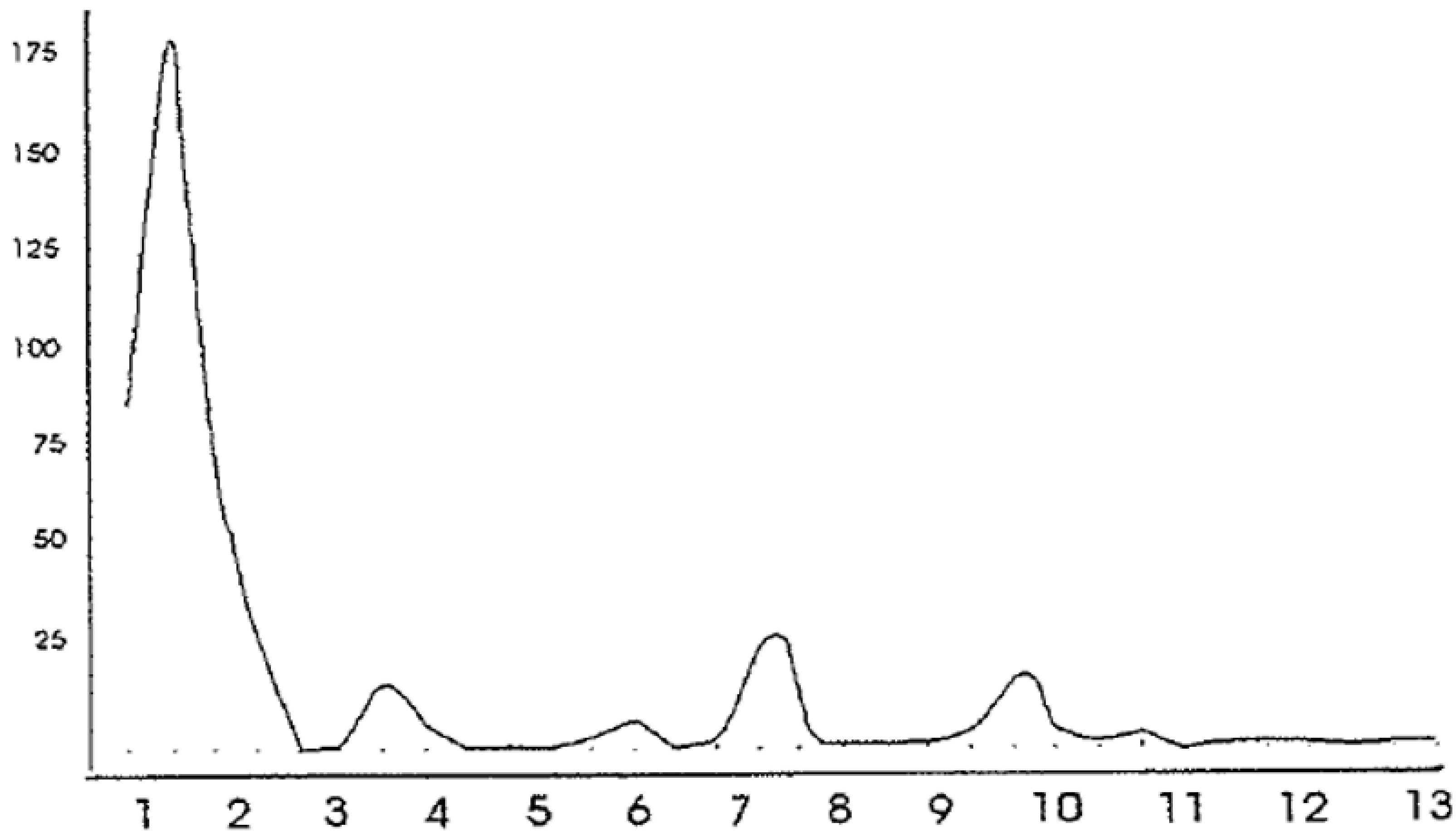
f(t)=10sin(2π ω)+5sin(10π ω)+2cos(30π ω)



Magnitude spectrum of f(t)

Making sense from squiggles……

PSD of a rat EEG



Ad absurdum

Any periodic waveform can be represented by a Fourier series

Assume the skyline of Manhattan is periodic……..

In Figure 5-4a, the series

- 80 -20.31cos(ω) -11.42sin(ω)

is graphed from 0 to 2π radians. In Figure 5-4b, the first harmonic,

- 20.18cos(2ω) +23.5sin(2ω)

is added. In Figure 5-4c, the series is extended to include three more harmonics:

- 0 cos(3ω)-10.6 sin(3ω)-.3cos(4ω)+9.26 sin5ω) +5.48cos(5ω)-1.24 sin(5ω).

The Manhattan skyline is drawn in Figure 5-4d created from the series:

- the fundamental ω, a DC offset of 80, and 511 harmonics.

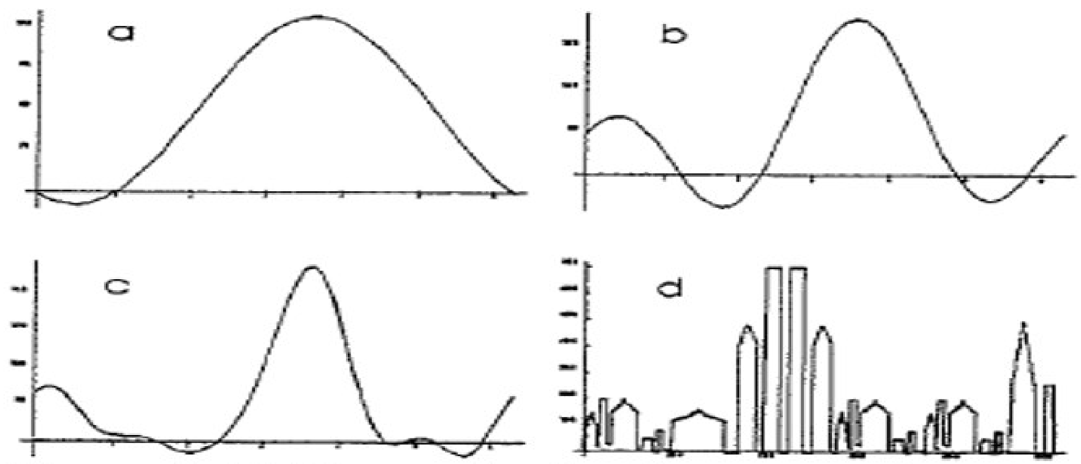
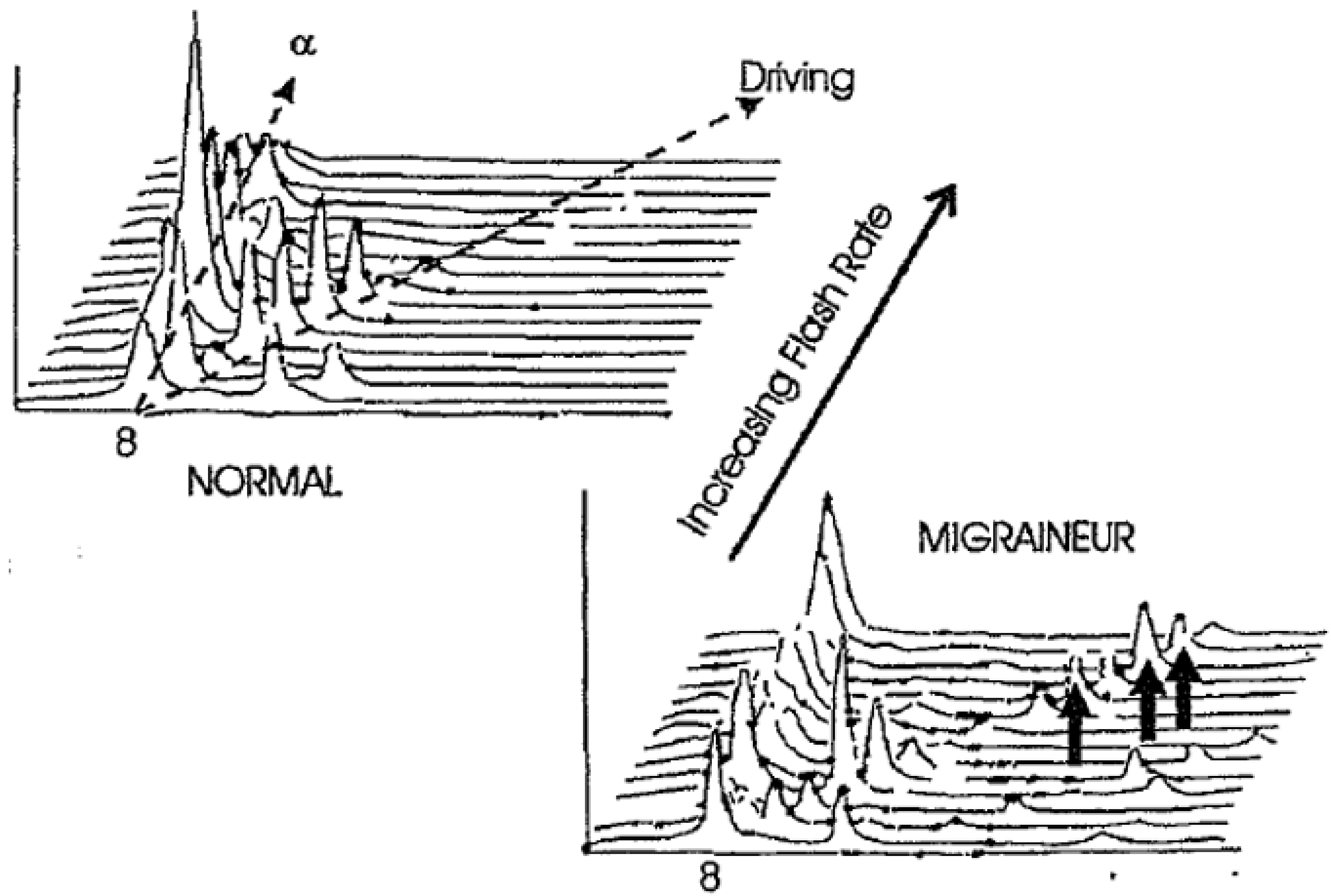


Figure 5-42 The synthesis of an assumed-repeating skyline, summed from 512 components of sines and cosines of harmonics, weighted by their Fourier coefficients. You can see that the series still has not converged by looking at the construction of the sloping rooftops.

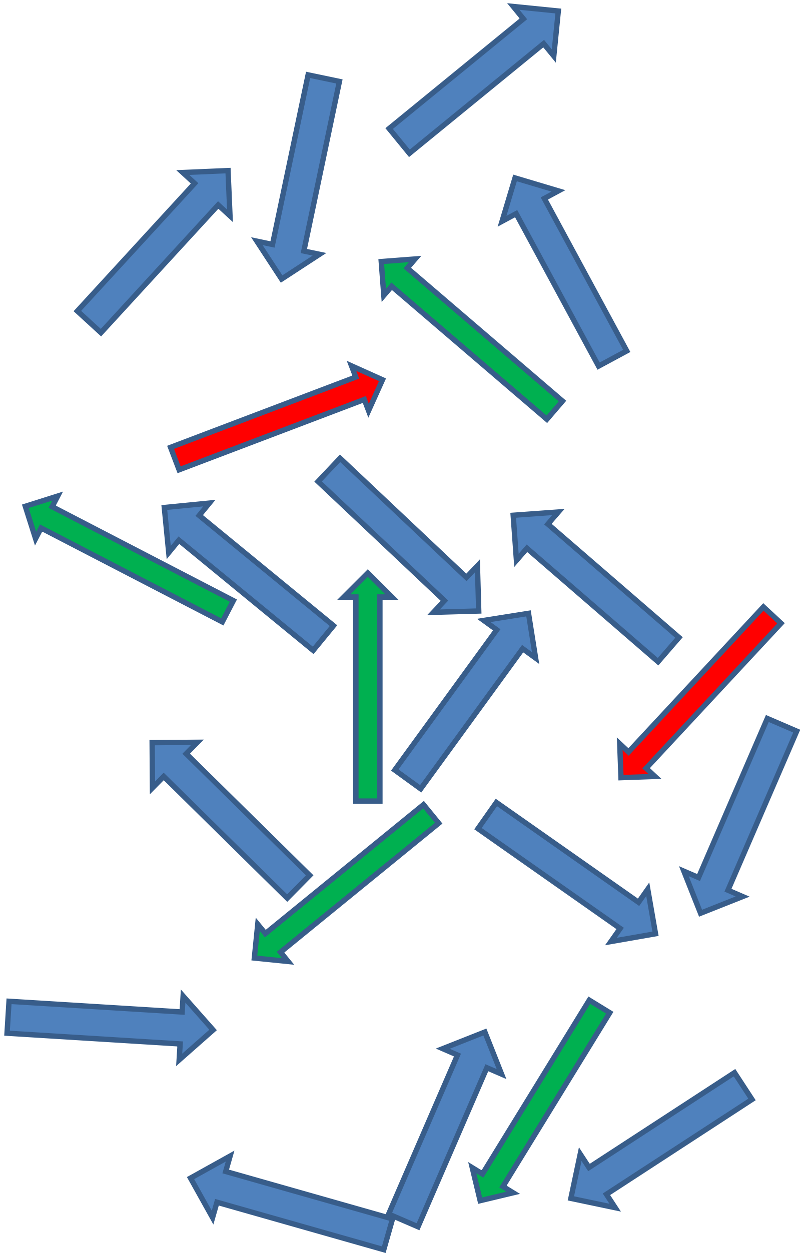
Information nearly impossible to glean from the time series



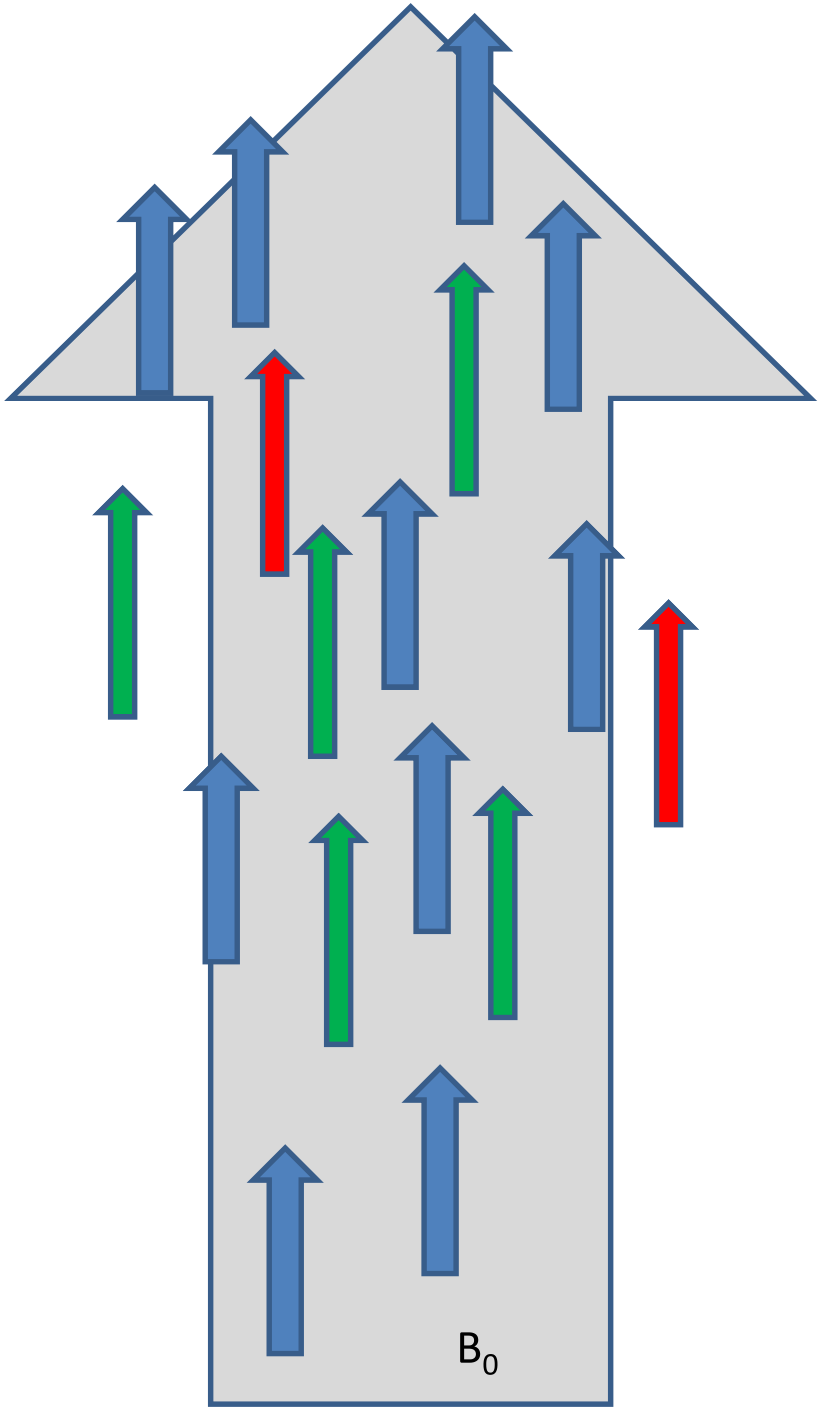
PSDs of migraineurs showing enhanced photic driving in the first harmonic of each driving frequency (z-axis) as well as increasing power in the α-band (7-9*hz*)

Example of Fourier Transformation in Action

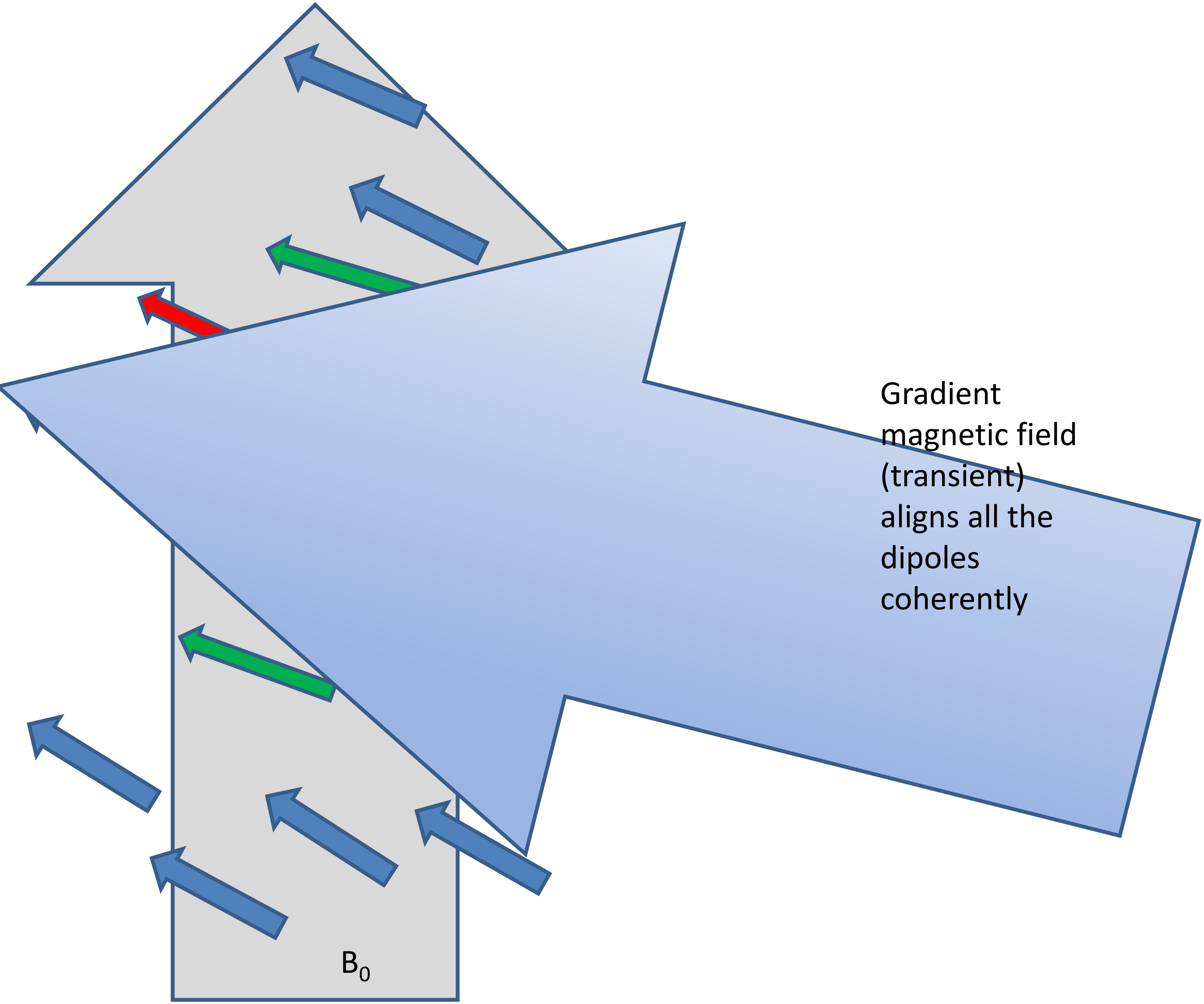
Magnetic Resonance Imaging

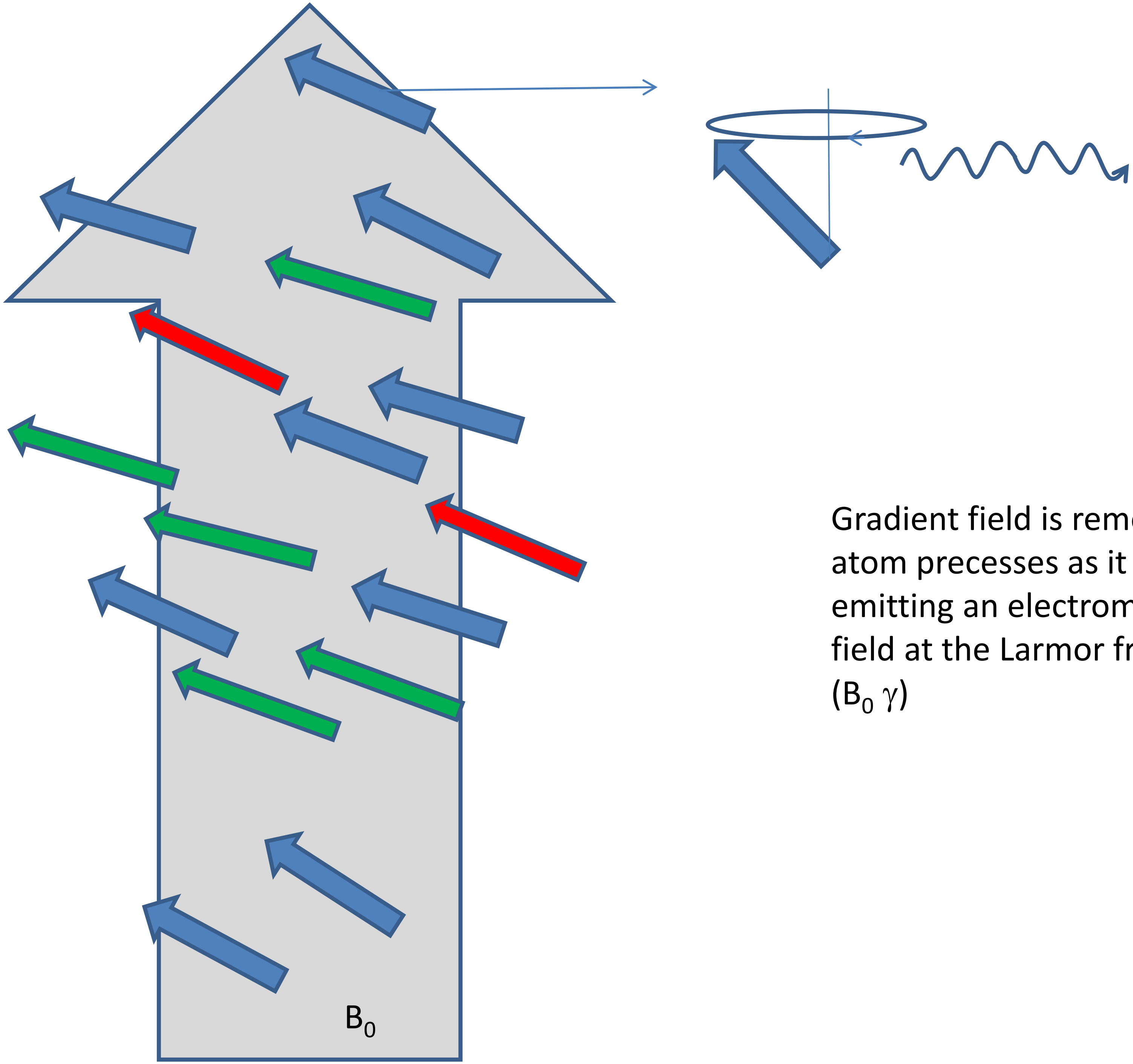


3 species, each with its own gyromagnetic ratio γ

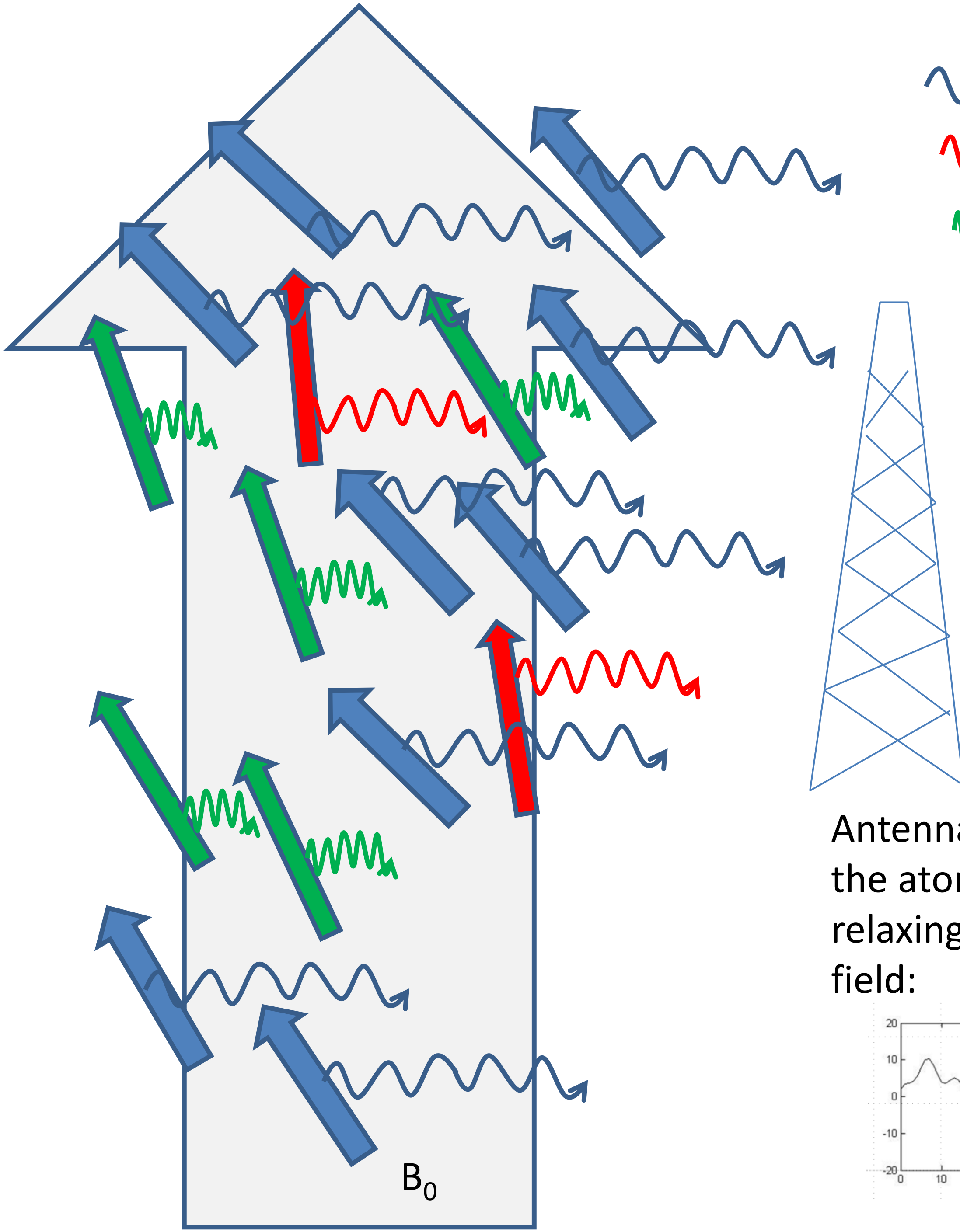


Species align in a stable magnetic field B_0

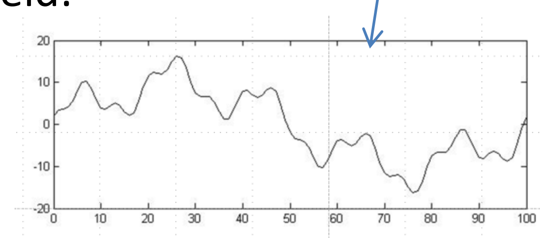

Gradient magnetic field (transient) aligns all the dipoles coherently
B_0



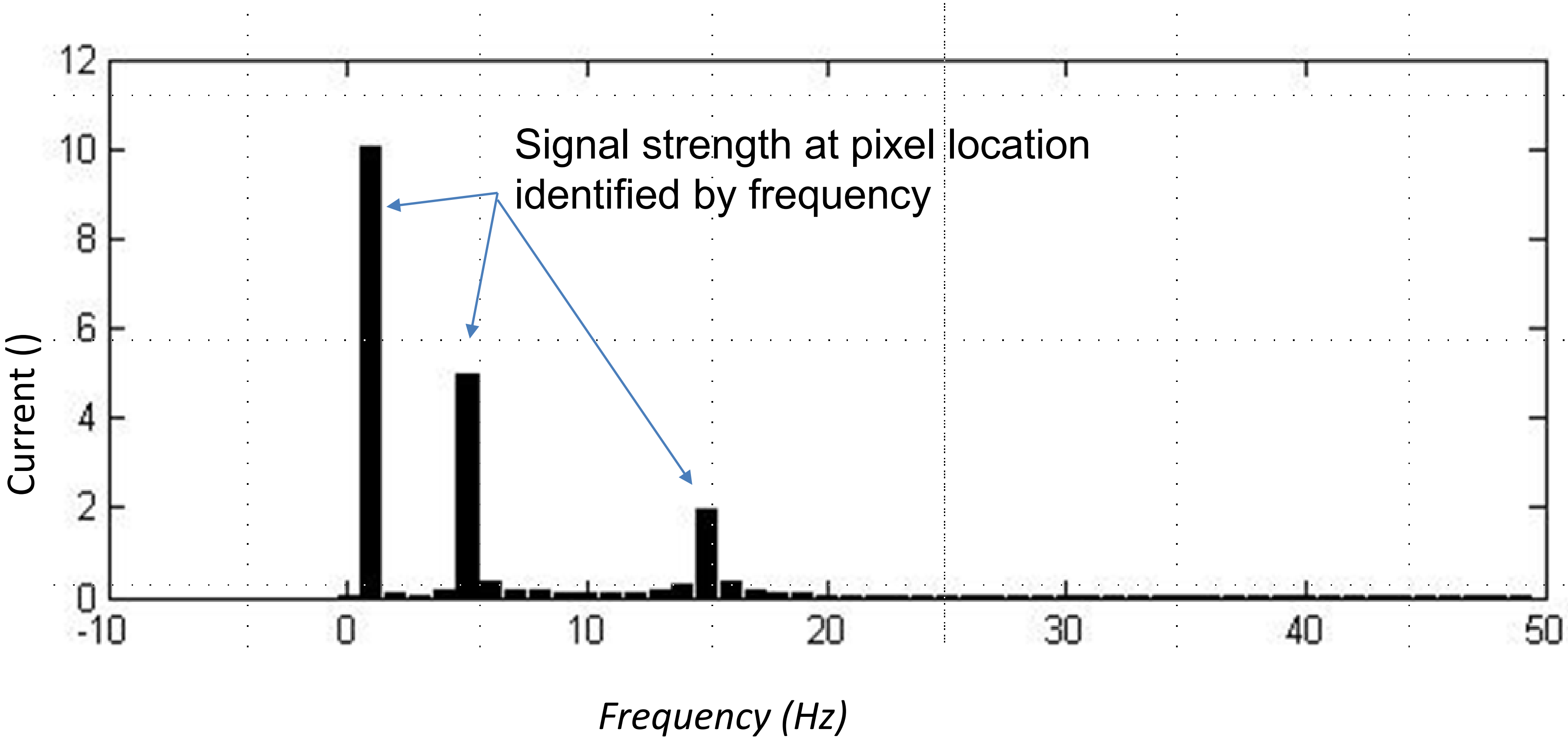
Gradient field is removed. Each atom precesses as it relaxes, emitting an electromagnetic field at the Larmor frequency (B_0 γ)



Antenna receives this signal as the atoms precess while relaxing to realign in the B_0 field:



Here is the Magnitude spectrum of the Fourier Transformed signal at the antenna



Pixel locations

Applications of Power Spectra

- Autospectrum
- Cross Spectrum
- Coherence

$$Coherence = \frac{\left|F_{xy}\right|^2}{F_{xx}F_{yy}}$$

- Autocorrelation
- Cross Correlation

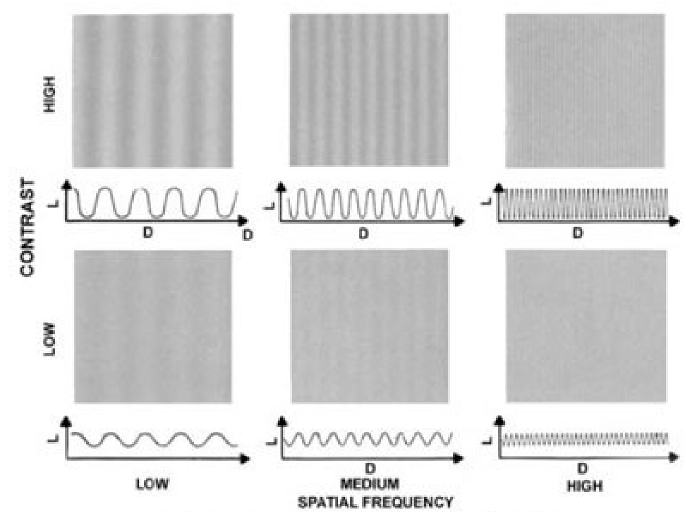
2 Dimensional Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty}\int f(x,y)e^{-i2\pi(ux+vy)}dxdy$$

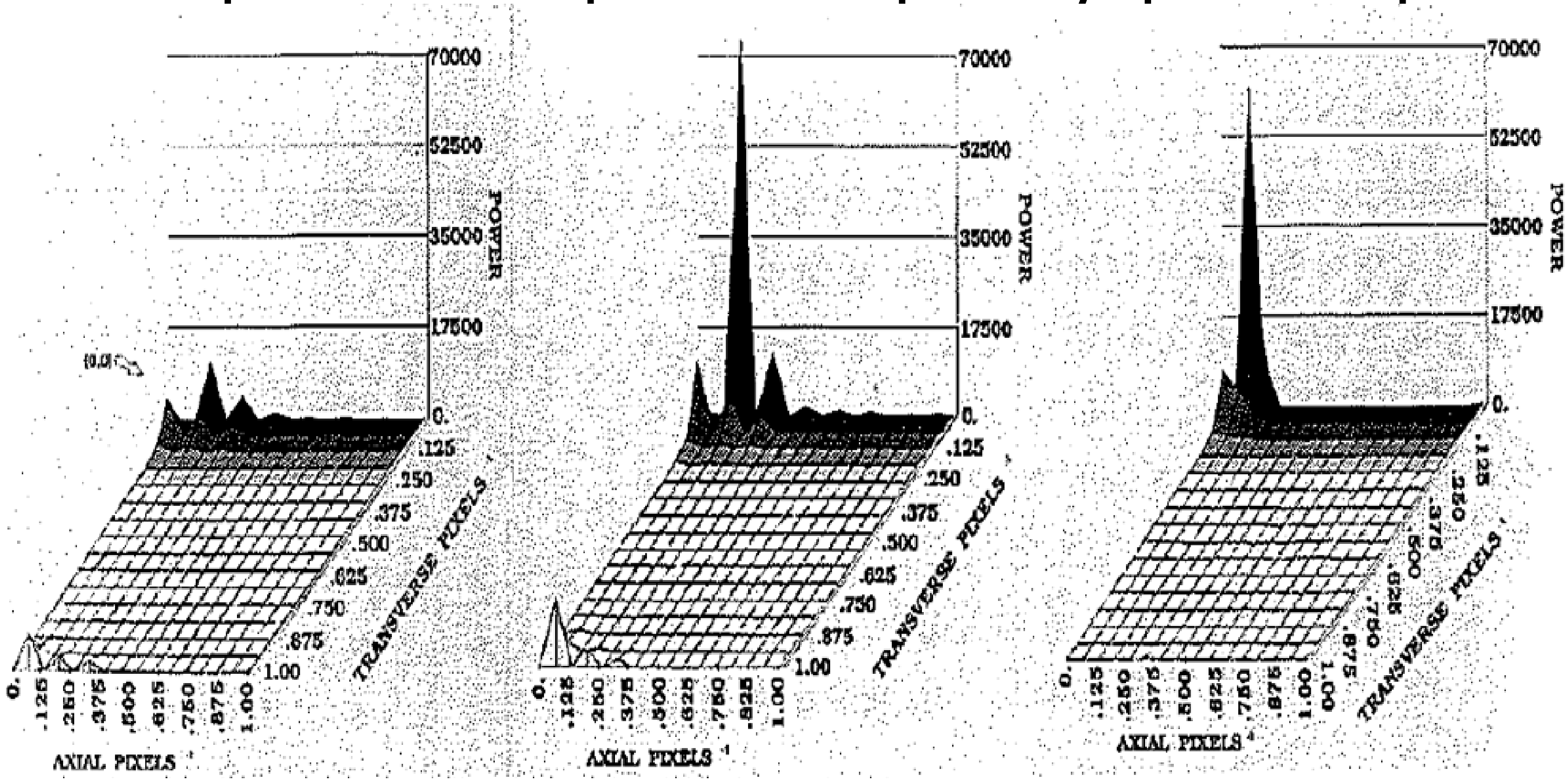
Spatial Frequency

The Fourier transform can be applied to any periodic function. This need not be a function of time but can be a function of density (or intensity or energy *etc*) across space. Where frequency is sec^{-1}, spatial frequency could be, say, m^{-1}, or perhaps $A^{\circ -1}$

1-D spatial frequency demonstration



An example of 2-D spatial frequency power spectrum



Analysis of images of microbubbles aggregating in a glioma over time (left to center) and mature glioma without microbubble contrast (right).

Of the 16 spatial frequencies and 256 pixels, eight were found to have strong discriminating power, identifying contrast and no contrast in tumors, *vis à vis* other 'bright' objects

Many, many applications

- The assembly of data scattered from a perfect crystal can be represented by a Fourier transform. Theoretically, the inverse transform would reveal the underlying structure leading to the observed scatter pattern.
- Alas, there are no phase data, only magnitude, so an inverse transform is impossible
- But there are workarounds…
- Much more on this in a lecture to come.

Convolution

The convolution integral:

$$f * g = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)$$

Flip *g* to -τ
Offset *g* by t
Integrate the product of *f* and *g*

Convolution

- Filtration in the time domain
- Convolution theorem
  - Convolution in the time domain =Multiplication in the frequency domain
  - Multiplication in the time domain=convolution in the frequency domain
- Leakage..multiplication in the time domain with a noncontinuous function

Digital issues

- Sampling
  - Aliasing
  - Nyquist frequency
  - Sample impulse convolution