Discrete Homotopy and Homology Theory for Metric Spaces

Hélène Barcelo
Mathematical Sciences Research Institute

Algebraic and Combinatorial Approaches
in Systems Biology — UConn
May 23, 2015

Overview

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2. $\mathcal{A}_n(\Gamma, v_0)$ - set of graph homs $f : \mathbb{Z}^n \to V(\Gamma)$, that is,

   if $d(\vec{a}, \vec{b}) = 1$ in $\mathbb{Z}^n$ then $d(f(\vec{a}), f(\vec{b})) = 0$ or 1, with
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   $f(\vec{i}) = v_0$ almost everywhere
3. f, g are *discrete homotopic* if there exist $h \in \mathcal{A}_{n+1}(\Gamma, v_0)$ and $k, \ell \in \mathbb{N}$ such that for all $\vec{i} \in \mathbb{Z}^n$,
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4. $A_n(\Gamma, v_0)$ - set of equivalence classes of maps in $\mathcal{A}_n(\Gamma, v_0)$
   Note: translation preserves discrete homotopy

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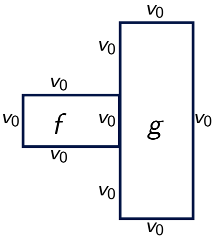

$$[f\,g] = [f]\,[g]$$

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f :

1	r	a	h	n	i	e	R
0	n	e	b	u	a	L	d
-1	60	r	e	h	c	a	b
	-3	-2	-1	0	1	2	3

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$$f : \begin{array}{r|ccccccc} 1 & r & a & h & n & i & e & R \\ 0 & n & e & b & u & a & L & d \\ -1 & 60 & r & e & h & c & a & b \\ \hline & -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \qquad f^{-1} : \begin{array}{r|ccccccc} 1 & R & e & i & n & h & a & r \\ 0 & d & L & a & u & b & e & n \\ -1 & b & a & c & h & e & r & 60 \\ \hline & -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}$$

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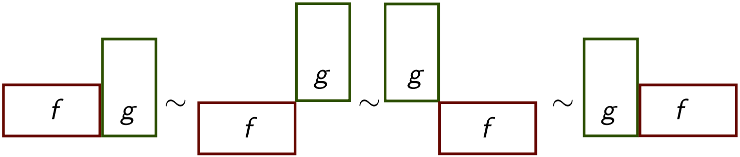

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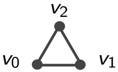
$A_1\Big($ 

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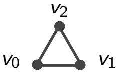
$A_1\Big($ 

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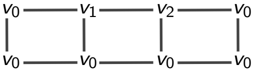

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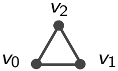
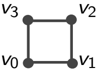
$A_1($  $, v_0) = 1$

$A_1($  $, v_0) = 1$

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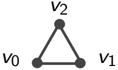
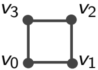
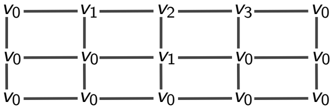
$A_1\Big($  $, v_0\Big) = 1$

$A_1\Big($  $, v_0\Big) = 1$



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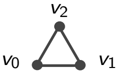
$A_1\Big($ 

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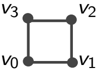
$A_1\Big($ 

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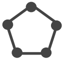
$A_1\Big($  $, v_0\Big) \cong \mathbb{Z}$

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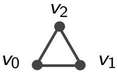
$A_1\Big($ 

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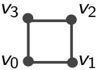
$A_1\Big($ 

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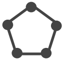
$A_1\Big($  $, v_0\Big) \cong \mathbb{Z}$

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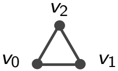
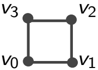
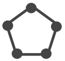
$A_1\Big($  $, v_0\Big) = 1$

$A_1\Big($  $, v_0\Big) = 1$

$A_1\Big($  $, v_0\Big) \cong \mathbb{Z}$

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(2-dim cell complex: attach 2-cells to $\triangle$, $\square$ of Γ)

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- $A_n^q(\Delta, \sigma_0) \cong A_n(\Gamma_\Delta^q, \sigma_0)$

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- $A_n^r(X, x_0)$ r-Lipschitz maps $f : \mathbb{Z}^n \to X$ (stabilizing in all directions)

$$f : X \to Y \text{ is } r\text{-Lipschitz} \iff d(f(x_1), f(x_2)) \leq r\, d(x_1, x_2)$$

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   - Degenerate singular n-cube: if $\exists\, i$ s.t. $(A_i^n\sigma) = (B_i^n\sigma)$

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    - $D_n(\Gamma) :=$ free abelian group generated by all degenerate singular n-cubes

Discrete Homology Theory for Graphs

(B., Capraro, White)

Necessities

1. Discrete n-dim cube $Q_n = \{(a_1, \ldots, a_n) \mid a_i = 0 \text{ or } 1\}$
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$$C_n(\Gamma) := \mathcal{L}_n(\Gamma)/D_n(\Gamma)$$

elements of C_n correspond to n-chains

Discrete Homology Theory for Graphs

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$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i \left(A_i^n(\sigma) - B_i^n(\sigma) \right)$$

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The *relative homology* groups of (Γ, Γ'):

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A. Mayer-Vietoris sequence:

$$\cdots \xrightarrow{\partial_*} DH_n(\Gamma_1 \cap \Gamma_2) \xrightarrow{\text{diag}} DH_n(\Gamma_1) \oplus DH_n(\Gamma_2) \xrightarrow{\text{diff}} DH_n(\Gamma_1 \cup \Gamma_2) \xrightarrow{\partial_*} DH_{n-1}(\Gamma_1 \cap \Gamma_2) \quad \cdots$$

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- For each abelian group G and $\bar{n} \in \mathbb{N}$, there is a finite connected simple graph Γ such that

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- There is a graph S^n such that

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Unexpected Application of Discrete Homotopy Theory

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Complex $K(\pi, 1)$ Spaces

$\mathcal{A}^{\mathbb{C}}_{n,2}$ braid arrangement:
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$\pi_1(M(W_{n,3})) \cong \mathrm{Ker}(\phi')$
where $W_{n,3}$ is a 3-parabolic
subgroup of type W
(B-Severs-White 2009)

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3. Pure braid gp: $\mathrm{Ker}(\phi)$, where ϕ: "$S_n -$ (i)" $\rightarrow S_n$ by $\phi(s_i) = s_i$

$$\pi_1(M(\mathcal{A}_{n,2}^{\mathbb{C}})) \cong \mathrm{Ker}(\phi)$$

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5. $W' =$"$W - \{$(iii),(iv)$,\dots\}$" $\implies \pi_1(M(W_{n,3})) \cong \mathrm{Ker}(\phi')$

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B. Intersect S^{n-1} with hyperplane arrangement $W \rightsquigarrow$ simplicial decomposition of S^{n-1}

$$\Delta_0 \text{ the 3-parabolic subspace arrangement of type } W$$
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- X is the subcomplex of the W-permutahedron gotten by removing the faces corresponding to Δ_0, i.e. removing the faces bounded by 6-cycles, 8-cycles,...

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E.

$$\text{Ker}(\phi)/N \cong \text{Ker}(\phi')$$

where $\phi'\colon$ "$W - \{$(iii),(iv),$\ldots\}$"$\to W$ by $\phi(s_i) = s_i$

We have replaced a group (π_1) defined in terms of the topology of a space with a group (A_1) defined in terms of the combinatorial structure of the space.